A Dynamical Model of Water Recycling in a Mine-Processing Enterprise[1]

Richard F. Hartl
University of Vienna, Department of Business Administration
Brünnerstr. 72, A-1210 Vienna, Austria
e-mail: richard.hartl@univie.ac.at

Gagik T. Kirakossian,
Department of Computer Systems & Informatics
State Engineering University of Armenia
Terian str. 105, 375009, Yerevan
e-mail: gkirakos@arminco.com

Stanislav R. Hakobyan
State Engineering University of Armenia
Terian str. 105, 375009, Yerevan
e-mail: slava@freenet.am

Artyom H. Markaryan
State Engineering University of Armenia
Terian str. 105, 375009, Yerevan
e-mail: markart@freenet.am

Abstract: We consider a mine-processing enterprise, which runs a number of production plants. Each plant uses a considerable amount of water which is polluted during the production process. The aim is to determine the optimal amounts of water to be taken from a river or to be pumped back from a waste water reservoir. The problem is decomposed in two tasks. Task 1 is formulated as a time-discrete optimal control model and determines the amounts of water pumped in each period. The solution method used is dynamic programming. While Task 1 has an ecological objective (minimize environmental damage), in Task 2 the aim is to determine, by which pump configurations the desired quantities of water are most efficiently brought to reservoir (with least cost). Because of the special structure of this linear programming problem it can be solved simply by sorting with respect to unit costs.

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1 Introduction

This paper deals with a problem that is motivated by the operative planning in an Armenian mine-processing enterprise; see e.g. Kirakossian (1999). This firm runs a number of production plants. Each plant uses a considerable amount of water which is polluted during the production process and is then led into a waste water reservoir, where sand and other solid waste is deposited on the ground. There is also a number of smaller water reservoirs from which the plants can take the water. These water reservoirs can either be replenished by fresh water from a nearby river or by pumping back waste water from the waste water reservoir. The aim is to determine the optimal amounts of water to be taken from the river and from a waste water reservoir along with the decision on how much water each production plant takes from which of the smaller reservoirs. On the one hand there is a different cost associated with each source of water, and on the other hand, at each reservoir the deviation from desired levels of water should be minimized. These desired levels represent some kind of safety stock for coping with unexpected events (e.g. breakdowns of pumps).

The structure of the water recycling problem is given in Figure 1 where the parameters and variables mentioned will be used in the mathematical models.

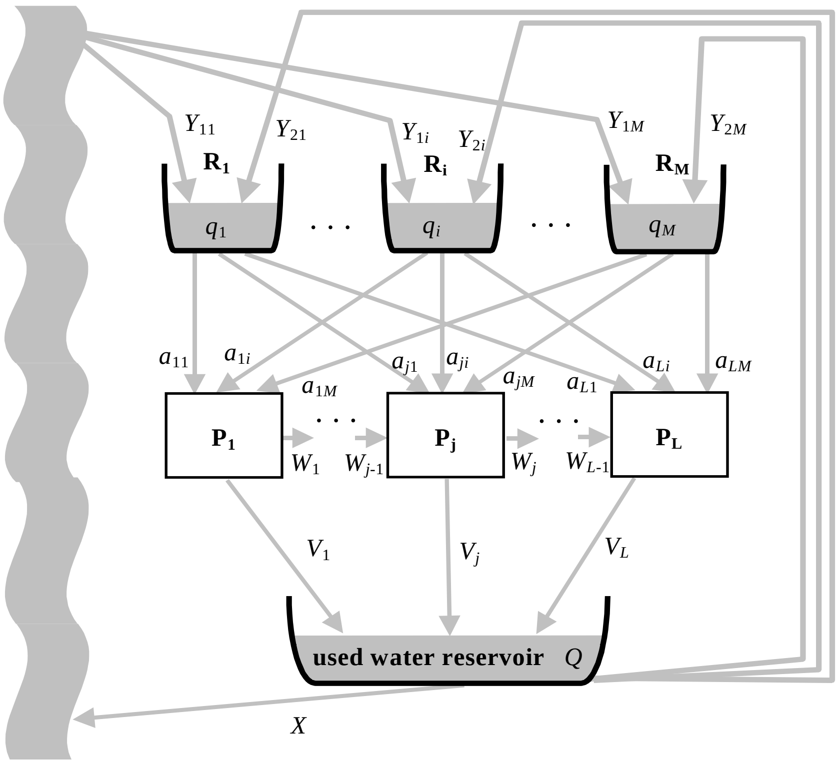

Fig. 1. The structure of water recycling in a mining enterprise

The first step of the problem is the synthesis of optimal dynamic control for providing the maximum use of water resource wastes within 24 hours of a continuous manufacturing process. The purpose is also to determine the optimal working time of each pump configuration within any time interval meeting all of the water-supply structure constraints so that the maximum effect according to the chosen optimization criterion is provided.

2 Model Formulation for Task 1

The first task of the overall planning process is to determine how much water should be pumped to each reservoir from the river and from the waste water reservoir, and how much water should be pumped from the waste water reservoir to the river. The cost structure is mainly ecological. There are linear costs associated with the amounts of pumped water and quadratic deviation costs associated with the deviations from desired levels in the reservoirs.

The planning period for the mine-processing enterprise is twenty four hours. This period is divided into *N* time intervals, where in each of these time intervals the pump configuration that supplies water (either primary, i.e., fresh water or secondary, i.e., used water) to a reservoir (or to several reservoirs) must not change. Such a pump configuration can consist of pumps (set of pumps or pump stations) of different types (and different in number), but also a configuration in which no pump station is active is possible.

In the water-supply structure of the technological process there are *M* water reservoirs. They can be replenished by two different sources of water, primary water and waste water. For $i = \overline{1, M}$, $n = \overline{1, N}$ let

$Y_{1i}(n)$ … amount of fresh primary water (for example from a nearby river) pumped into reservoir *i* in interval *n*, and

$Y_{2i}(n)$ … amount of used and filtered water from a storage reservoir of used water pumped into reservoir *i* in interval *n*.

$q_i(n)$ … level of water in reservoir *i* by the end of time interval *n*

For each reservoir *i* minimum and maximum water levels $q_i^{\min}$ and $q_i^{\max}$ must be observed. Furthermore, lower and upper limits are given also for the water flows to each reservoir from the river, $Y_{1i}^{\min}$ and $Y_{1i}^{\max}$, and the waste water reservoir, $Y_{2i}^{\min}$ and $Y_{2i}^{\max}$. Also given are the desired levels of water by the end of each time interval:

$\hat{q}_i(n)$ … desired level of water in reservoir *i* by the end of time interval *n.*

The initial level $Q(0)$ of water in the used water reservoir is given. In order not to exceed the upper limit of the waste water reservoir, water can be pumped from there into the river:

$X(n)$ … amount of used water pumped from the waste water reservoir into the river in interval *n.*

The production process consists of L technological sub-processes, each of which is performed in a separate production plant (mill) which can be considered a consumer of water. For $i=\overline{1,M}$, $j=\overline{1,L}$, and $n=\overline{1,N}$ let:

$a_{ji}(n)$ … amount of water pumped from reservoir i to consumer (plant) j in interval n.

$V_j(n)$ … volume of water pumped from consumer j to the storage reservoir of waste water in time interval n.

$W_{j-1}(n)$ … volume of water pumped from consumer j-1 to consumer j.

These quantities $a_{ji}(n)$, $V_j(n)$, and $W_{j-1}(n)$ are given and are determined by technological requirements or by some production planning that precedes the planning steps to be considered here. However, the following model could also handle a situation where e.g. $V_j(n)$, and $W_{j-1}(n)$ can be varied.

See also Figure 1 for an illustration of the various flows of water.

With the above parameters, the decision variables $Y_{1i}(n)$, $Y_{2i}(n)$, and $X(n)$ (flow variables), and $q_i(n)$ and $Q(n)$ (stock variables) the mathematical models of **Task 1** can be formulated as follows:

$$\boldsymbol{min}\sum_{i=1}^{M}\sum_{n=1}^{N}\left[Y_{1i}(n)H_{1i}+Y_{2i}(n)H_{2i}+X(n)H_3+\left(q_i(n)-\hat{q}_i(n)\right)^2R_i\right], \tag{1}$$

subject to

$$q_i(n)=q_i(n-1)+Y_{1i}(n)+Y_{2i}(n)-\sum_{j=1}^{L}a_{ji}(n),\ i=\overline{1,M},\ n=\overline{1,N}, \tag{2}$$

$$Q(n)=Q(n-1)+\sum_{j=1}^{L}V_j(n)-\sum_{i=1}^{M}Y_{2i}(n)-X(n),\ n=\overline{1,N}, \tag{3}$$

$$0=\sum_{i=1}^{M}a_{ji}(n)+W_{j-1}(n)-W_j(n)-V_j(n),\ n=\overline{1,N},\ j=\overline{1,L}, \tag{4}$$

$$Q(0)=Q^0,\ q_i(0)=q_i^0,\ i=\overline{1,M}, \tag{5}$$

$$q_i^{\min}\leq\ q_i(n)\leq q_i^{\max},\ n=\overline{1,N},\ i=\overline{1,M}, \tag{6}$$

$$Y_{1i}^{\min}\leq\ Y_{1i}(n)\leq Y_{1i}^{\max},\ n=\overline{1,N},\ i=\overline{1,M}, \tag{7}$$

$$Y_{2i}^{\min}\leq\ Y_{2i}(n)\leq Y_{2i}^{\max},\ n=\overline{1,N},\ i=\overline{1,M}, \tag{8}$$

$$Q^{\min}\leq\ Q(n)\leq Q^{\max},\ n=\overline{1,N}, \tag{9}$$

$$0\leq X(n)\leq X^{\max}\ n=\overline{1,N}, \tag{10}$$

where the cost parameters H_{1i}, H_{2i}, H_3, and R_i are positive quantities and can be determined experimentally. Note that cost parameter R_i should not be mixed up with the notation $\mathbf{R_i}$ for reservoir *i* in Figure 1.

In the objective function (1), we have linear costs associated with the flows of water which can e.g. be seen as the environmental damage caused by using energy to operate the pumps. In some cases, if the flow is downwards, no pumps may be necessary. Then the associated unit costs are very small or even zero. This holds in particular for the flow from the used water reservoir to the river, but there may be some other environmental costs H_3 of leading waste water into the river. On the other hand, deviations from desired levels of the reservoirs are – as usual – penalized by quadratic costs.

Equations (2) are the balance equations for the small water reservoirs where the inflows from the river are $Y_{1i}(n)$ and the inflows from the waste water reservoir are $Y_{2i}(n)$, while the total outflows to the *L* plants are $\sum_{j=1}^{L} a_{ji}(n)$. Equation (3) is the balance equation for the waste water reservoir where the total inflow from the *L* plants is $\sum_{j=1}^{L} V_j(n)$ and the total outflow consists of the outflows $\sum_{i=1}^{M} Y_{2i}(n)$ pumped to the *M* reservoirs and the amount of used water pumped into the river $X(n)$. Equations (4) make sure that the amount of water flowing into any plant (consumer) *j* equals the total outflow from this plant. Here, the total inflow consists of the inflows $\sum_{i=1}^{M} a_{ji}(n)$ from the *M* reservoirs and the inflow from the preceding consumer, $W_{j-1}(n)$; the total outflow consists of the flow $W_j(n)$ towards the following plant in the line and the outflow towards the waste water reservoir, $V_j(n)$. The initial conditions are provided by (5) while (6) – (9) summarize the lower and upper bounds for the stock and flow variables, respectively.

Apparently, the nonlinear programming problem (1) – (10) has a time structure w.r.t. the time intervals *n*, and therefore can be solved using dynamic programming. Clearly, the problem could also be formulated as an optimal control problem in continuous time; see e.g. Feichtinger and Hartl (1986). However, because data are usually available in discrete time, we have chosen the discrete time formulation.

Let us now discuss the possible outcomes of optimization, the flow variables $Y_{1i}(n)$, $Y_{2i}(n)$ and $X(n)$, depending on the values of the parameters involved. In particular, we are interested in the sensitivity w.r.t. the cost parameters H_{1i}, H_{2i}, and H_3 if there is a sufficient level of water in the storage reservoir of wastes for meeting the production needs.

Proposition 1. *If the constraints* (9) *on the waste water reservoir are not binding (which is usually the case in reality) and if the desired levels* q_{ci} are constant over time, *then there are three typical cases:*

a) *If* $H_{1i} > H_{2i}$ *, i.e., if the water from the river is more expensive than the water from the waste water reservoir, then it will be optimal to take water for the production needs from the waste water reservoir mainly.*

In particular, if also the upper bound of flow constraint (8) *is not binding, then:*

$$Y_{2i}(n) = \sum_{j=1}^{L} a_{ji}(n) - Y_{1i}^{\min} \text{ and } Y_{1i}(n) = Y_{1i}^{\min}.$$

while reservoir i is kept at the desired level $\hat{q}_i$ *.*

If there is a demand peak for reservoir i and the upper bound of flow constraint (8) *is binding, then it is optimal to use the cheapest resource at maximum level:*

$$Y_{2i}(n) = Y_{2i}^{\max} \text{ and } Y_{1i}(n) = \sum_{j=1}^{L} a_{ji}(n) - Y_{2i}^{\max} - \varepsilon$$

where ε *is the amount by which the water level in reservoir i decreases in this time interval.*

b) *If* $H_{1i} = H_{2i}$ *, then the optimal solution is not unique since taking water from the primary and secondary sources incurs the same costs. This is a hairline case.*

c) *If* $H_{1i} < H_{2i}$ *, then it will be optimal to take water for the production needs from the primary water sources (river) mainly.*

In particular, if also the upper bound of flow constraint (7) *is not binding, then only water from the river will be used (except for the minimum value* $Y_{2i}^{\min}$ *of used water in case this is positive):*

$$Y_{1i}(n) = \sum_{j=1}^{L} a_{ji}(n) - Y_{2i}^{\min} \text{ and } Y_{2i}(n) = Y_{2i}^{\min}.$$

while reservoir i is kept at the desired level $\hat{q}_i$ *.*

If there is a demand peak for reservoir i and the upper bound of flow constraint (7) *is binding, then it is optimal to use the cheapest resource at maximum level:*

$$Y_{1i}(n) = Y_{1i}^{\max} \text{ and } Y_{2i}(n) = \sum_{j=1}^{L} a_{ji}(n) - Y_{1i}^{\max} - \varepsilon$$

where ε *is the amount by which the water level in reservoir i decreases in this time interval.*

Proof and Discussion: The proof of these results is straightforward and can be formalized by applying the usual Kuhn-Tucker conditions to (1) – (10) and using the complementary slackness conditions, saying that multipliers must vanish if the corresponding constraint is not binding. The economic interpretation is also clear.

The reason why the water level in reservoir *i* can decrease (at a small rate ε) if the upper bound on the cheapest resource is binding is the following: taking all the excess demand from the more expensive source while keeping the water level in reservoir *i* constant (at its desired level) causes unit extra costs of $|H_{1i} - H_{2i}|$; on the other hand, taking a small quantity ε out of the stock is almost cost free because of the quadratic cost $(q_i(n) - \hat{q}_i(n))^2 R_i$ which is $\varepsilon^2 R_i$ when starting from the desired level; see also Section 4.

When there is a demand peak for reservoir *i*, then it might be optimal to start increasing the level in reservoir *i* beyond its desired level some periods before, letting it fall below this desired level during the demand peak and to fully replenish it later. This resembles standard results from the theory of production smoothing under convex costs; see e.g. Arrow and Karlin (1958) or Hartl (1995).

As usual, these results will be somewhat modified by end effects for time intervals *n* close to the terminal time *N*. In particular, for *n* = *N* it will no longer be optimal to keep the reservoirs at their desired level. Rather, it will always be optimal to reduce the stocks a little at the end (with hardly any extra costs because of the quadratic cost structure) thus saving some (linear) costs of replenishment. Since these undesired end effects always occur in multi period problems it is as usual reasonable to apply planning on a rolling basis, i.e., to implement only the optimal solution for the first few periods and then to reoptimize the system. □

Taking into account the objective (1), the results of this analysis indicate that the cost parameter H_{2i} should be a small positive number, and H_{1i} and R_i should be large positive quantities so as to provide a small quantity of $Y_{1i}(n)$ and a big quantity of $Y_{2i}(n)$ as a result of optimizing (1) – (10) with a minimum deviation of $q_i(n)$ from $q_{ci}(n)$.

As mentioned, problem (1) – (10) is a standard (time structured) nonlinear programming problem, which could be solved using some standard NLP package. One could also exploit the similarity to *quadratic network control problems* (QNCP); see e.g. Ernst and Goh (2000).

On the other hand, because of the time structure the problem is also well suited for being solved using dynamic programming. This is the approach we choose here; see Section 4.

3 Model Formulation for and Solution of Task 2

The optimal solution of the time-discrete optimal control model (1) – (10) gives values for all the decision variables $Y_{1i}(n)$, $Y_{2i}(n)$, and $X(n)$ (flow variables) and $q_i(n)$ and $Q(n)$ (stock variables). The stock variables as well as the flow variables where no pumps are involved are not relevant for Task 2. The optimal values of $Y_{1i}(n)$ and $Y_{2i}(n)$ are denoted by $Y_{1i}^*(n)$ and $Y_{2i}^*(n)$ for all reservoirs $i = \overline{1, M}$ and time intervals $n = \overline{1, N}$.

These are input data for the second task, namely to determine, by which pump configurations the desired quantities of water $Y_{1i}^{*}(n)$ and $Y_{2i}^{*}(n)$ are most efficiently brought to reservoir *i* from the river and the waste water reservoir, respectively. While Task 1 had an ecological objective (minimize environmental damage), in Task 2 we have an economic objective (minimize cost).

A pump configuration *s* is an aggregate of pumps (of the same or of different type) that pump water from a certain source to a certain technological point (i.e. in the direction of concrete reservoirs). Let us assume that the necessary quantity to be pumped is Y^*, which can either be $Y_{1i}^{*}(n)$ or $Y_{2i}^{*}(n)$ for some reservoir *i* and some time interval *n*. Task 2 is a static optimization problem and has to be solved at most 2*MN* times, namely for every time interval *n*, for every reservoir *i* and for every source (k = 1 or 2) where $Y_{ki}^{*}(n) > 0$; see also Figure 2.

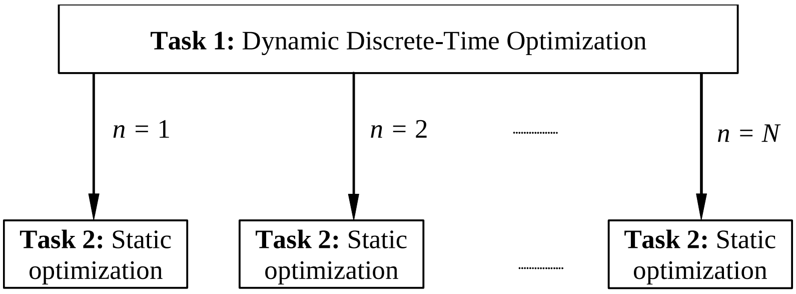

Fig. 2. The hierarchically structured optimization problem.

The number of pump configurations working from the chosen source to the chosen target location is *S*. The number of operating hours (in the chosen time interval *n*) of pump configuration *s* is denoted by Z_S, while in the time interval considered, at most *T* operating hours are available. The volume of water pumped through pump configuration *s* per unit time (within one hour of work) is given by P_S and the operating cost of configuration *s* per unit time is C_S. Thus, the total water volume pumped by configuration *s* is $P_S Z_S$ and the total cost from operating configuration *s* is $C_S Z_S$.

The objective function is to minimize total operating cost:

$$\boldsymbol{min} \sum_{s=1}^{S} C_S Z_S \tag{11}$$

subject to the constraints that the required amount of water Y^* is pumped:

$$\sum_{s=1}^{S} P_S Z_S = Y^*, \tag{12}$$

$$0 \le Z_S \le T\ . \tag{13}$$

This is a special case of a linear programming problem which can be solved quite straightforward using the following simple algorithm:

1. Resort the configurations such that $C_1/P_1 \le C_2/P_2 \le \cdots \le C_S/P_S$.
2. Set s = 1 and Y = 0
3. IF $P_s T > Y^* - Y$ THEN $Z_s = \dfrac{Y^* - Y}{P_s}$; STOP.

   ELSE $Z_s = T$; set $s = s + 1$; and repeat step 3.

This will insure that always those pump configurations are used, where the cost per unit of water, C_s/P_s , are as small as possible.

4 Solution of Task 1

As mentioned in Section 2 we will solve Task 1 (1) – (10) by dynamic programming. For this, we first solve the *one period problem* (time interval *N*) for all possible given initial water levels $q(N-1) = [q_1(N-1), \cdots, q_M(N-1)]$ and $Q(N-1)$.

$$F\big(q(N-1), Q(N-1), N\big) = \boldsymbol{min} \sum_{i=1}^{M} \Big[Y_{1i}(N) H_{1i} + Y_{2i}(N) H_{2i} + X(N) H_3 + \big(q_i(N) - \hat{q}_i(N)\big)^2 R_i \Big], \quad (14)$$

subject to

$$q_i(N) = q_i(N-1) + Y_{1i}(N) + Y_{2i}(N) - \sum_{j=1}^{L} a_{ji}(N) , \; i = \overline{1, M} , \quad (15)$$

$$Q(N) = Q(N-1) + \sum_{j=1}^{L} V_j(N) - \sum_{i=1}^{M} Y_{2i}(N) - X(N) , \quad (16)$$

$$0 = \sum_{i=1}^{M} a_{ji}(N) + W_{j-1}(N) - W_j(N) - V_j(N) , \; j = \overline{1, L} , \quad (17)$$

$$q_i^{\min} \le q_i(N) \le q_i^{\max} , \; i = \overline{1, M} , \quad (18)$$

$$Y_{1i}^{\min} \le Y_{1i}(N) \le Y_{1i}^{\max} , \; i = \overline{1, M} , \quad (19)$$

$$Y_{2i}^{\min} \le Y_{2i}(N) \le Y_{2i}^{\max} , \; i = \overline{1, M} , \quad (20)$$

$$Q^{\min} \le Q(N) \le Q^{\max} , \quad (21)$$

$$X(N) \ge 0 . \quad (22)$$

Apparently (after elimination of the stock variables $q_i(N)$ and $Q(N)$) this is the static nonlinear optimization problem in the variables $Y_{1i}(N)$, $Y_{2i}(N)$, $V_j(N)$, and $X(N)$:

$$F\big(q(N-1), Q(N-1), N\big) = \\ \boldsymbol{min} \sum_{i=1}^{M} \Big[Y_{1i}(N) H_{1i} + Y_{2i}(N) H_{2i} + X(N) H_3 \\ + \left(q_i(N-1) + Y_{1i}(N) + Y_{2i}(N) - \sum_{j=1}^{L} a_{ji}(N) - \hat{q}_i(N) \right)^2 R_i \Big] ,$$

subject to

$$0 = \sum_{i=1}^{M} a_{ji}(N) + W_{j-1}(N) - W_j(N) - V_j(N) , \ \ j = \overline{1, L} ,$$

$$q_i^{\min} \le \ \ q_i(N-1) + Y_{1i}(N) + Y_{2i}(N) - \sum_{j=1}^{L} a_{ji}(N) \le q_i^{\max} , \ \ i = \overline{1, M} ,$$

$$Y_{1i}^{\min} \le \ \ Y_{1i}(N) \le Y_{1i}^{\max} , \ \ i = \overline{1, M} ,$$

$$Y_{2i}^{\min} \le \ \ Y_{2i}(N) \le Y_{2i}^{\max} , \ \ i = \overline{1, M} ,$$

$$Q^{\min} \le \ \ Q(N-1) + \sum_{j=1}^{L} V_j(N) - \sum_{i=1}^{M} Y_{2i}(N) - X(N) \le Q^{\max} ,$$

$$X(N) \ge 0 ,$$

which could be solved analytically by exploiting the Kuhn-Tucker conditions.

The result of solving the one period problem (14) – (22) is the *value function* $F\big(q(N-1), Q(N-1), N\big)$ assigning to each initial vector $q(N-1)$ and each initial value of $Q(N-1)$ the associated cost in the last interval $n = N$.

Then, we solve the *two period problem* (time intervals $N - 1$ and N) for all possible given initial water levels $q(N-2) = \big[q_1(N-2), \cdots, q_M(N-2)\big]$ and $Q(N-2)$:

$$F\big(q(N-2), Q(N-2), N-1\big) = \\ \boldsymbol{min} \sum_{i=1}^{M} \Big[Y_{1i}(N-1) H_{1i} + Y_{2i}(N-1) H_{2i} + X(N-1) H_3 \\ + \big(q_i(N-1) - \hat{q}_i(N-1)\big)^2 R_i \Big] , \\ + F\big(q(N-1), Q(N-1), N\big) \qquad (23)$$

subject to constraints similar to (15) – (22) where simply the indiced are adjusted from N to *N-1*.

This problem could hardly be solved analytically anymore, since there are too many cases already.

The result of solving the two period problem is the *value function* $F\left(q(N-2), Q(N-2), N-1\right)$ assigning to each initial vector $q(N-2)$ and each initial value of $Q(N-2)$ the associated cost in the last two intervals $n = N - 1$ and $n = N$.

Next, the *three period problem* (time intervals $N - 2$, $N - 1$ and N) must be solved for all possible initial water levels $q(N-3) = \left[q_1(N-3), \cdots, q_M(N-3)\right]$ and $Q(N-3)$, which gives the *value function* $F\left(q(N-3), Q(N-3), N-2\right)$, and so on.

Finally, we arrive at the *N period problem* (time intervals 1 to N) where for the given initial water levels $q(0) = \left[q_1(0), \cdots, q_M(0)\right]$ and $Q(0)$, the *value function* $F\left(q(0), Q(0), 1\right)$ gives the optimal cost for the global problem (1) – (10).

5 Numerical Example

An application program *WaterFlowOptimizer* has been developed, which automatically performs the calculations according to the given algorithm.

To illustrate this algorithm we provide a numerical example. As real data are considered confidential, we use artificial data for illustration. The data concerning the value of water resources are measured by a unit of 1000 cubic meters. The main parameters defining the dimension of the model are:

number of time intervals	$N = 5$
number of water tanks	$M = 10$
number of consumers	$L = 10$
initial water volume in a reservoir	$q_i(0) = 100$

and the given water flows that cannot be influenced are given in Tables 1 and 2:

	$V_j(n)$									
$n \setminus j$	1	2	3	4	5	6	7	8	9	10
1	1.57	2.14	2.71	3.29	3.86	4.43	5.00	5.57	6.14	6.71
2	2.14	2.71	3.29	3.86	4.43	5.0	5.57	6.14	6.71	7.29
3	2.71	3.29	3.86	4.43	5.00	5.57	6.14	6.71	7.29	7.86
4	3.29	3.86	4.43	5.00	5.57	6.14	6.71	7.29	7.86	8.43
5	3.86	4.43	5.00	5.57	6.14	6.71	7.29	7.86	8.43	9.00

Table. 1. $V_j(n)$ – volume of water pumped from consumer j to the storage reservoir of waste water in time interval n.

$W_{j\text{-}1}(n)$									
$n \setminus j$	2	3	4	5	6	7	8	9	10
1	1.71	2.43	3.14	3.86	4.57	5.29	6.00	6.71	7.43
2	2.43	3.14	3.86	4.57	5.29	6.00	6.71	7.43	8.14
3	3.14	3.86	4.57	5.29	6.00	6.71	7.43	8.14	8.86
4	3.86	4.57	5.29	6.00	6.71	7.43	8.14	8.86	9.57
5	4.57	5.29	6.00	6.71	7.43	8.14	8.86	9.57	10.29

Table. 2. $W_{j\text{-}1}(n)$ – volume of water pumped from j-1 to consumer j in time interval n

For each time interval, the water volumes going from the reservoirs to the consumers as well as minimal and maximal water volumes for every reservoir are also given, but we refrain from giving these values here.

Tables 3 and 4 present the optimal solution obtained by the above DP approach:

$Y_{1i}(n)$										
$n \setminus i$	1	2	3	4	5	6	7	8	9	10
1	1.33	1.56	1.75	1.90	2.03	2.14	2.24	2.32	2.40	2.47
2	1.33	1.56	1.75	1.90	2.03	2.14	2.24	2.32	2.40	2.47
3	1.67	1.95	2.18	2.38	2.54	2.68	2.80	2.91	3.00	3.08
4	2.00	2.34	2.62	2.85	3.04	3.22	3.36	3.49	3.60	3.70
5	2.00	2.34	2.62	2.85	3.04	3.22	3.36	3.49	3.60	3.70

Table. 3. The amount of fresh primary water pumped into reservoir i in interval n.

$Y_{2i}(n)$										
$n \setminus i$	1	2	3	4	5	6	7	8	9	10
1	2.67	2.44	2.25	2.10	1.97	1.86	1.76	1.68	1.60	1.53
2	2.67	2.44	2.25	2.10	1.97	1.86	1.76	1.68	1.60	1.53
3	3.33	3.05	2.82	2.63	2.46	2.32	2.20	2.09	2.00	1.92
4	4.00	3.66	3.38	3.15	2.96	2.78	2.64	2.51	2.40	1.92
5	4.00	3.66	3.38	3.15	2.96	2.78	2.64	2.51	2.40	2.30

Table. 4. The amount of used and filtered water from the storage reservoir pumped into reservoir i in interval n.

The cost data H_{1i} and H_{2i} reflect the fact, that R_1 is more close to the used water reservoir and R_M is more close to the river. Hence, the reservoirs with high indiex numbers mainly use fresh water while the reservoirs with a smaller index mainly use recycled water; see also Figure 3 for the first time interval. If the data were chosen in a way that resembles Figure 1 the opposite result would be obtained.

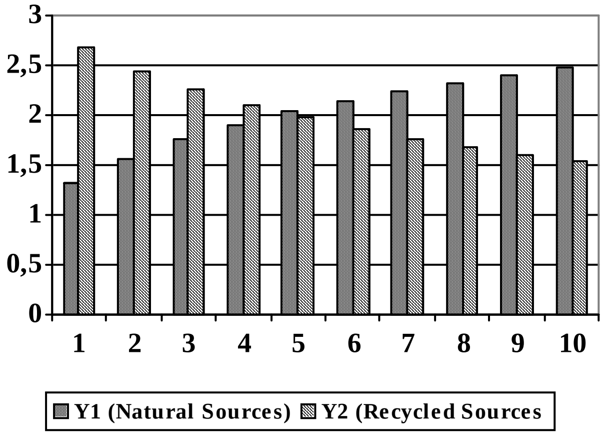

Fig. 3. The use of fresh and recycled water to replenish the 10 reservoirs in the first time interval.

Another observation is that over time the use of fresh and recycled water increases. This means that initially the comparatively high water levels $q_i(0)$ in the 10 reservoirs is reduced and in later periods, when the level has already decreased, more water has to be pumped into the reservoirs.

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