Proof for Theorem 3

Proof. Let us consider a feasible solution of the $GLSP^{PM}_{ToolSync}$. Due to the subtour elimination guay by Christian Almeder and Bernardo Almada-Lobo, there exists for each period $t$ a well defined sequence of tools $\Phi_{tm} : k_{tln1} \rightarrow k_{tln2} \cdots \rightarrow k_{tln\lambda_m}$ of length $\lambda_m$ that are attached to machine $m$. Such a sequence contains each tool at most once, except the last tool which might be identical with the first tool ($k_{tln1} = k_{tln\lambda_m}$). These sequences are determined by the values of $\alpha_{tmk}$ and $T_{tmkl}$, i.e. if one of the variables are changed, the corresponding sequence changes and vice versa. From now on, we omit the indices $t$ and $m$ of $k_{tln}$ and $\lambda_m$ for an easier representation.

We have to set the number of micro-periods in the $GLSP^{PM}_{ToolSync}$ to $R \geq K + 1$ (note that $\lambda \leq K + 1$) and define now a mapping $f$ as follows (for an easy representation all variables of the $GLSP^{PM}_{ToolSync}$ not appearing in that mapping are set to 0):

\[
\begin{pmatrix}
I_{it} \\
B_{it} \\
X_{itmk} \\
\alpha_{tmk} \\
T_{tmkl} \\
\mu_{tmk}^s \\
\mu_{tmk}^f \\
\mu_{tmmax} \\
\Phi_{tm}
\end{pmatrix} \mapsto
\begin{pmatrix}
I_{it} \\
B_{it} \\
x_{it1mk_1} = X_{itmk_1}, x_{it2mk_2} = X_{itmk_2}, \ldots, x_{it(\lambda-1)mk_{\lambda-1}} = X_{itmk_{\lambda-1}}, \\
0 &= 0, k_\lambda \neq k_1, \\
x_{it\lambda mk_\lambda} = \begin{cases}
X_{itmk_\lambda} & k_\lambda \neq k_1 \\
0 & k_\lambda = k_1
\end{cases} \\
y_{t1mk_1} = y_{t2mk_2} = \cdots = y_{t\lambda mk_\lambda} = y_{t(\lambda+1)mk_\lambda} = \cdots = y_{tRmk_\lambda} = 1 \\
z_{t1mk_1k_1} = z_{t2mk_2k_2} = z_{t3mk_3k_3} = \cdots = z_{t\lambda mk_\lambda k_\lambda} = 1, \\
z_{t(\lambda+1)mk_\lambda k_\lambda} = \cdots = z_{tRmk_\lambda k_\lambda} = 1 \\
u_{t1m} = \mu_{tmk_1}^s, \ldots, u_{t(\lambda-1)m} = \mu_{tmk_{\lambda-1}}^s, u_{tAm} = \mu_{tmmax}^s - s_{mk_{\lambda-1}k_\lambda}, u_{t(\lambda+1)m} = \cdots = u_{tRm} = 1 \\
u_{t1m} = \mu_{tmk_1}^f, \ldots, u_{t(\lambda-1)m} = \mu_{tmk_{\lambda-1}}^f, u_{tAm} = \cdots = u_{tRm} = 1
\end{pmatrix}
\]

According to the sequence of tools $\Phi_{tm}$ of the $GLSP^{PM}_{ToolSync}$ solution, the first $\lambda$ micro-periods of the $GLSP^{PM}_{ToolSync}$ are used each for the production with the corresponding tool. During the remaining micro-periods no tool exchange is performed and the length of those micro-periods is set to zero ($u_{trm}^s = u_{trm}^f = 1, r > \lambda$). All decision variables of the $GLSP^{PM}_{ToolSync}$ appearing in active constraints are used in the mapping, i.e. if any of those variables changes in a feasible way, also the mapped solution of the $GLSP^{PM}_{ToolSync}$ changes. Hence, the mapping is injective.

Now we show that all constraints of the $GLSP^{PM}_{ToolSync}$ are fulfilled. The inventory balance constraints hold, because the $I_{it}$ and $B_{it}$ are not changed, and the total production during a
macro-period for each item is the same in both model formulations

\[
\sum_{m=1}^{M} \sum_{r=1}^{R} \sum_{k=1}^{K} x_{itrmk} = \sum_{m=1}^{M} \sum_{r=1}^{R} \sum_{k=1}^{K} X_{itmk} - \alpha_{tmk} \cdot X_{itmk} = \sum_{m=1}^{M} \sum_{k=1}^{K} X_{itmk}.
\]

Due to constraints (20) and (37) of the CLSP\textsubscript{ToolSync} also the constraints (15) and (16) of the GLSP\textsubscript{ToolSync} hold and because of the construction of the the mapping also (12), (13) and (14) are fulfilled and we can state that \(z_{tmk} = T_{tmk} = 1\) and all other \(T_{tmkl} = 0\).

Because of constraints (28) also constraints (3) are fulfilled. If we consider constraints (26) and (24) we may conclude that

\[
\mu_{tmk}^s + \sum_{i=1}^{N} \mu_{tmk}^s \cdot X_{itmk} + \sum_{l=1}^{K} \left( s_{mlk} T_{tmk} + s_{mk} T_{tmkl} - s_{mk} T_{lmk} \right) - s_{mk} T_{lmk} = \mu_{tmk}^s \ \forall t, m < \lambda - 1
\]

holds for all subsequent tools in the path \(\Phi_{tm}\) except for the last two tools \(k_{\lambda-1}\) and \(k_{\lambda}\). Hence, constraints (4) are fulfilled for all \(r < \lambda\), respectively also for all \(r > \lambda + 1\) because all start times for those last micro-periods are 1 and there are no production and no tool switches. Because of (22)-(30) and (26) the following constraints can be derived for all \(t\) and \(m\)

\[
u_{t}^{\text{max}}_{tm} = \mu_{tm}^{f} - s_{mk_{\lambda-1}}k_{\lambda} = \mu_{tmk_{\lambda-1}}^{s} - s_{mk_{\lambda-1}}k_{\lambda} \geq 0
\]

\[
u_{t}^{s}_{tmk_{\lambda-1}} + \sum_{i=1}^{N} \mu_{tmk_{\lambda-1}}^{s} \cdot X_{itmk_{\lambda-1}} + \sum_{l=1}^{K} s_{mlk} T_{tmk_{\lambda-1}} = \nu_{t}^{s}_{t_{(\lambda-1)}m} + \sum_{i=1}^{N} \mu_{tmk_{\lambda-1}}^{s} \cdot X_{itmk_{\lambda-1}} + s_{mk_{\lambda-2}}k_{\lambda-1}z_{t_{(\lambda-1)}m}k_{\lambda-1}
\]

\[
u_{t}^{f}_{t_{(\lambda+1)}m} = 1 \geq \mu_{tm}^{max} \geq \mu_{tmk_{\lambda-1}}^{f} - s_{mk_{\lambda-1}}k_{\lambda} \geq 0
\]

\[
u_{t}^{s}_{tmk_{\lambda-1}} + \sum_{i=1}^{N} \mu_{tmk_{\lambda-1}}^{s} \cdot X_{itmk_{\lambda-1}} + \sum_{l=1}^{K} s_{mlk} T_{tmk_{\lambda-1}} = \nu_{t}^{s}_{t_{(\lambda+1)}m} + \sum_{i=1}^{N} \mu_{tmk_{\lambda-1}}^{s} \cdot X_{itmk_{\lambda-1}} + s_{mk_{\lambda-2}}k_{\lambda-1}z_{t_{(\lambda+1)}m}k_{\lambda-1}
\]

Hence, constraints (4) hold also for \(r = \lambda\) and \(r = \lambda + 1\).

Constraints (5) are valid, because if we rewrite (24) for two subsequent tools \(k_{\nu}\) and \(k_{\nu+1}\) (except the last two tools) of the path \(\Phi_{tm}\), we will get for all \(t\) and \(m\)

\[
u_{t}^{f}_{tmk_{\nu}} \leq \nu_{t}^{s}_{tmk_{\nu+1}} + s_{mk_{\nu+1}}k_{\nu+1} \cdot T_{tmk_{\nu}}k_{\nu+1} = T_{tmk_{\nu}}k_{\nu+1} + 1 + \alpha_{(t+1)mk_{\nu+1}} = \nu_{t}^{f}_{tmk_{\nu+1}} + s_{mk_{\nu+1}}k_{\nu+1}
\]

So (5) hold for \(r < \lambda - 1\) and because of constraints (24) they are fulfilled for \(r = \lambda\). For \(r > \lambda\) the relations hold because \(u_{t_{(r+1)}m} = u_{t_{rm}} = 1\) and \(\sum_{k=1}^{K} s_{mk} \cdot z_{t_{(r+1)}m} = 0\). Constraints (5)
are valid, because either \( \lambda < R \) and \( u^s_{tRm} = u^f_{tRm} = 1 \) and \( x_{itRmk} = 0 \) or \( \lambda = R \) and constraints (32)–(34) and (27) ensure the necessary relations between \( u^s_{tRm} \) and \( u^f_{tRm} \). Constraints (7) and (8) follow directly from (24)–(27), (32)–(34) and the definition of the mapping \( f \).

In the remaining part of the proof we will show that the synchronization constraints are valid. Constraints (11) together with (9) and (10) enforce, that either the start time of micro-period \( r \) on machine \( m \) has to be greater or equal than the finishing time of micro-period \( r' \) on a different machine \( m' \) if the same tool is used in that period or the other way around, depending which micro-period starts first. Since we only compare micro-periods where the same tools \( k \) are used, the mapping \( f \) identifies corresponding start and finishing times \( \mu^s_{tmk} \) and \( \mu^f_{tm'k} \) of the CLSP_PMToolSync. If tool \( k \) is the last tool on machine \( m \) and the considered micro-period is \( r = \lambda \), we have to take \( \mu^\text{max}_{tm} - \sum_s s_{m'lk} \cdot T_{tm'lk} \) as start time and 1 as finishing time. Because of constraints (30), (31), and (36) it is guaranteed, that the start time on one machine is always after the finishing time on another machine, or the other way around.

Finally, we show that the values of the objective functions are identical. Inventory and backorder levels are the same and as previously noticed \( z_{trmkl} = 1 \) only if a corresponding \( T_{tmkl} = 1 \) (for \( k \neq l \)). Since the \( z_{trmkl} \) values are determined by the path \( \Phi_{tm} \) and because of the subtour elimination no tool exchange can occur more than once, we can state that \( \sum_{r=1}^{R} z_{trmkl} = T_{tmkl} \) (for \( k \neq l \)). So together with the fact that \( s_{tmkk} = \sigma_{tmkk} = 0 \) it follows directly that also the setup cost parts of the objective functions are identical. \( \square \)