Optimal Investments with Convex-Concave Revenue: a Focus-Node Distinction

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Abstract

This paper considers a capital accumulation model in which revenue is a convex-concave function of the capital stock. While for certain capital values increasing returns to scale are reasonable, usually this property does not hold in general. In particular for large capital stock values the situation usually changes to decreasing returns to scale because it becomes increasingly difficult and thus expensive to produce more and more because of limitations of resources or infrastructure, lack of trained personnel in the region etc. We give a complete classification under which parameter constellations a saddle point equilibrium is optimal, under which parameter constellations it is optimal to close down by choosing zero investment and when history dependent equilibria occur. In the last scenario we distinguish between different types of the unstable equilibrium, which can each have their own implication for the firm's investment policy.

Key Words

Investment, Convex-concave revenue, Indeterminacy, Thresholds, Focus, Node

1 Introduction

In this paper we study a dynamic model of the firm with a standard capital accumulation framework. The firm's objective is to maximize the discounted cash flow stream. The firm's cash inflow consists of revenue and the cash outflow equals the cost of investment. Revenue is obtained by selling goods on the market. The firm needs a capital stock to produce these goods. The higher the capital stock, the more goods the firm produces, which in turn leads to a higher revenue. The firm can increase capital stock by investing. Technically spoken, we consider an optimal control model with one state variable, the capital stock, and one control variable, the investment rate.

The analysis of this framework goes back to the sixties, and started out with Eisner and Strotz (1963). In this contribution the revenue function was assumed to be concave and investment costs were convex. Using standard methods of control theory it is easily shown that optimal firm behavior prescribes convergence to a long run equilibrium at which marginal revenue equals marginal costs. Later it was recognized (Rothschild 1971) that arguments could be found in favor of a (partly) concave shape of the investment cost function. The problems (chattering controls!) that then occur in the maximization problem were subject of study in Davidson and Harris (1981) and Jorgensen and Kort (1993).

On the other hand it can also be the case that the revenue function is (partly) convexly shaped. Barucci (1998) studied the case where the revenue function is strictly convex throughout. He considers a framework where both the revenue function and the investment cost function are quadratic. As a result the isoclines, on which state, control, and co-state variables are constant, are linearly shaped, so that exactly one steady state exists. Barucci (1998) identifies a scenario where a saddle point equilibrium occurs for a positive level of the capital stock, and where convergence to this saddle point is the optimal policy.

From an economic point of view it seems that a strictly convex revenue function is hard to defend. The reason is that for large values of the capital stock it becomes increasingly difficult and thus more expensive to raise production due to limitations of resources or infrastructure, lack of trained personnel in the region, and so on and so forth. Moreover, increasing supply on the output market can cause a drop in price so that revenue increases less than proportionally with the stock of capital goods. Taking this argument into account, in this paper we want to study Barucci's framework, but then for a revenue function that is convex for low values of the capital stock and concave if capital stock is large. Departing from Barucci's quadratic specification, we obtained a convex-concave revenue function by simply considering a third order polynomial in which the third order argument is multiplied with a negative parameter. A capital accumulation model with a convex-concave revenue function was also studied in Dechert (1983) and Davidson and Harris (1981) (another related paper is Dechert and Nishimura 1983 but there a discrete Ramsey type framework is considered). From these contributions it can be concluded that partly convex revenue functions can lead to multiple equilibria. It then depends on the initial level of the capital stock to which of the equilibria it is optimal of the firm to converge to. In this sense we can speak of history dependent equilibria.

We were able to extend the results by Dechert (1983) and Davidson and Harris (1981) in

two ways. First, we can exactly identify the scenarios where different dynamic behavior occurs. In one scenario a unique saddle point arises to which the firm converges to in the long run. In two other scenarios it is optimal for the firm to converge to the origin. This is because in one scenario only steady states with negative capital stock values exist, while in the other scenario there is no steady state at all. Finally, one scenario exists with one stable and one unstable steady state. The stable steady state is a long run equilibrium, and it depends on the initial state whether it will be reached in the long run. For small firms it turns out to be optimal to refrain from investment and thus converge to the origin.
The second extension to Dechert (1983) and Davidson and Harris (1981) is that in the scenario with multiple equilibria we distinguish between different types of the unstable steady state. This unstable steady state can be a node or a focus and we show that this distinction can have consequences for the optimal dynamic investment policy of the firm. Furthermore we exactly identify for which parameter constellations this unstable steady state is a node or a focus.
The paper is organized as follows. Section 2 presents the model, while in Section 3 the necessary conditions for optimality are presented and the different scenarios are identified. Section 4 studies all scenarios except the one with multiple equilibria. This one is extensively analyzed in Section 5. Section 6 investigates the effect of an investment grant on optimal firm behavior.

2 Model formulation

The model we consider is the following:

$$\max_u \int_0^\infty e^{-\rho t}\left[r(k) - c(u)\right] dt; \qquad (1)$$

$$\dot{k} = u - \delta k; \quad k(0) = k_0; \qquad (2)$$

where k denotes the capital stock and u is investment. Labor is assumed to be proportional to capital stock so that it does not need to be explicitly included. The revenue function is given by $r(k)$ while the investment costs are $c(u)$. The discount rate is ρ while δ denotes the depreciation rate.
Although Barucci did not impose this constraint, for economic reasons (e.g. firm or industry specificity) we assume that investments are irreversible:

$$u \geq 0: \qquad (3)$$

While Barucci (1998) has assumed quadratic revenue and costs functions, we introduce a third order term in the revenue function in order to make it convex-concave:

$$r(k) = ak + bk^2 - fk^3; \quad c(u) = cu + du^2: \qquad (4)$$

We require all parameters a, b, c, d, f, δ, and ρ to be positive.

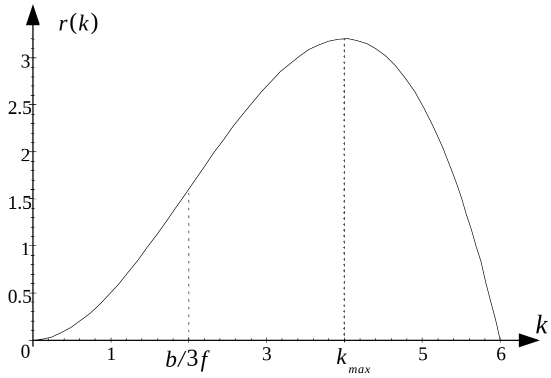

Figure 1: The revenue function $ak + bk^2 - fk^3$ for $a = 0, b = .6, f = 01$

Revenue $r(k)$ reaches its maximum for

$$k_{max} = \frac{b + \sqrt{b^2 + 3af}}{3f} \tag{5}$$

which is large if the coefficient b of the quadratic term is large and the coefficient f of the third order term is small. The relevant region of the problem is $0 \leq k \leq k_{max}$ since for larger k revenue decreases in k, and it is never optimal for the firm to be there.

The inflection point is

$$k = \frac{b}{3f}, \tag{6}$$

i.e., for $k < b/3f$ we have increasing returns to scale while for $k > b/3f$ there are decreasing returns to scale; see also Figure 1.

3 Mathematical Analysis

First the necessary conditions for optimality are presented, after which the equilibria are determined and their stability properties are studied.

3.1 Necessary conditions

To determine the necessary optimality conditions we first write down the current value Hamiltonian:

$$H = ak + bk^2 - fk^3 - cu - du^2 + q[u - \delta k]. \tag{7}$$

Since $H_{uu} < 0$, maximization of the Hamiltonian yields a unique solution. It follows that u is continuous over time (see Feichtinger and Hartl 1986, Corollary 6.2). From the maximum principle it is obtained that

$$\partial H/\partial u = 0 \quad \text{i.e.} \quad u = \frac{q - c}{2d}. \tag{8}$$

If (3) is imposed, then (8) holds only for $u > 0$ i.e. for $q > c$. Otherwise we have

$$u = 0 \quad \text{if} \quad q \leq c. \tag{9}$$

The adjoint equation is

$$\dot{q} = \rho q - \partial H/\partial k = (\rho + \mu)q - a - 2bk + 3fk^2. \tag{10}$$

From (8), i.e., $q = 2du + c$, and (10) we get:

$$\dot{u} = \frac{\dot{q}}{2d} = (\rho + \mu)u + \frac{(\rho + \mu)c - a}{2d} - \frac{b}{d}k + \frac{3fk^2}{2d}. \tag{11}$$

3.2 Equilibria and their Stability

We start with writing down the isoclines.
The $\dot{k} = 0$-isocline is

$$u = \mu k \tag{12}$$

and the $\dot{u} = 0$-isocline is

$$u = \frac{1}{2d(\rho + \mu)}[-3fk^2 + 2bk + a - (\rho + \mu)c]. \tag{13}$$

Hence, on the $\dot{u} = 0$-isocline u is a quadratic and concave function of k which has its maximum for

$$k = \frac{b}{3f} > 0. \tag{14}$$

For $k = 0$ the intersection with the u-axis occurs for

$$u = \frac{a - (\rho + \mu)c}{2d(\rho + \mu)},$$

which is the same value as in Barucci (1998), but the difference is that there the $\dot{u} = 0$-isocline is linear, while here it is quadratic.
From (12) and (13) we derive that equilibria must satisfy:

$$2[d(\rho + \mu)\mu - b]k + (\rho + \mu)c - a + 3fk^2 = 0.$$

Thus, the larger equilibrium is

$$\bar{k}_1 = \frac{(b - d(\rho + \mu)\mu) + \sqrt{(b - d(\rho + \mu)\mu)^2 + 3f(a - (\rho + \mu)c)}}{3f}, \quad \bar{u}_1 = \mu \bar{k}_1; \tag{15}$$

while the smaller equilibrium is given by

$$\bar{k}_2 = \frac{(b - d(\rho + \mu)\mu) - \sqrt{(b - d(\rho + \mu)\mu)^2 + 3f(a - (\rho + \mu)c)}}{3f}, \quad \bar{u}_2 = \mu \bar{k}_2. \tag{16}$$

From these expressions we can identify several scenarios which are stated in the following proposition.

Proposition 1 The four scenarios:

1. If $a > c(\rho+\delta)$ there exists a unique equilibrium in the relevant region ($k > 0; u > 0$; first quadrant), which is a saddle point. Furthermore the second equilibrium is unstable and located in the third quadrant.

2. If $a < c(\rho+\delta)$ and $f > \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ then no equilibria exists.

3. If $a < c(\rho+\delta)$; $f < \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ and $b > d(\rho+\delta)\delta$ then there exist 2 positive equilibria where the larger one is a saddle point and the other one is unstable.

4. If $a < c(\rho+\delta)$; $f < \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ and $b < d(\rho+\delta)\delta$ then there exist 2 negative equilibria (where the larger one is a saddle point and the other one is unstable).

Proof:

1. follows from the fact that for $a > c(\rho+\delta)$ the second term with the square root is always larger than the absolute value of the first term.

2. follows from the fact that for $a < c(\rho+\delta)$ and $f > \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ the term under the square root is negative.

3. follows from the fact that for $a < c(\rho+\delta)$; $f < \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ and $b > d(\rho+\delta)\delta$ the first term is positive and is always larger than the square root term.

4. follows from the fact that for $a < c(\rho+\delta)$; $f < \frac{(b - d(\rho+\delta)\delta)^2}{3((\rho+\delta)c - a)}$ and $b < d(\rho+\delta)\delta$ the first term is negative and its absolute value is always larger than the square root term.

The stability of the equilibria follows from the general observation that an equilibrium is a saddle point if and only if the derivative of the $\dot{k} = 0$-isocline is larger than the derivative of the $\dot{u} = 0$-isocline there:

$$\det \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial u} \\ \frac{\partial \dot{u}}{\partial k} & \frac{\partial \dot{u}}{\partial u} \end{bmatrix} < 0 \iff \left.\frac{du}{dk}\right|_{\dot{k}=0} = -\frac{\frac{\partial \dot{k}}{\partial k}}{\frac{\partial \dot{k}}{\partial u}} > -\frac{\frac{\partial \dot{u}}{\partial k}}{\frac{\partial \dot{u}}{\partial u}} = \left.\frac{du}{dk}\right|_{\dot{u}=0}$$

since $\frac{\partial \dot{k}}{\partial u} = 1 > 0$ and $\frac{\partial \dot{u}}{\partial u} = \rho + \delta > 0$. $\blacksquare$

We can also exploit equation (15) to obtain that both equilibria are always located on the increasing part of the revenue function:

Proposition 2 If $\bar{k}_1$ exists, then $\bar{k}_1 < k_{max}$

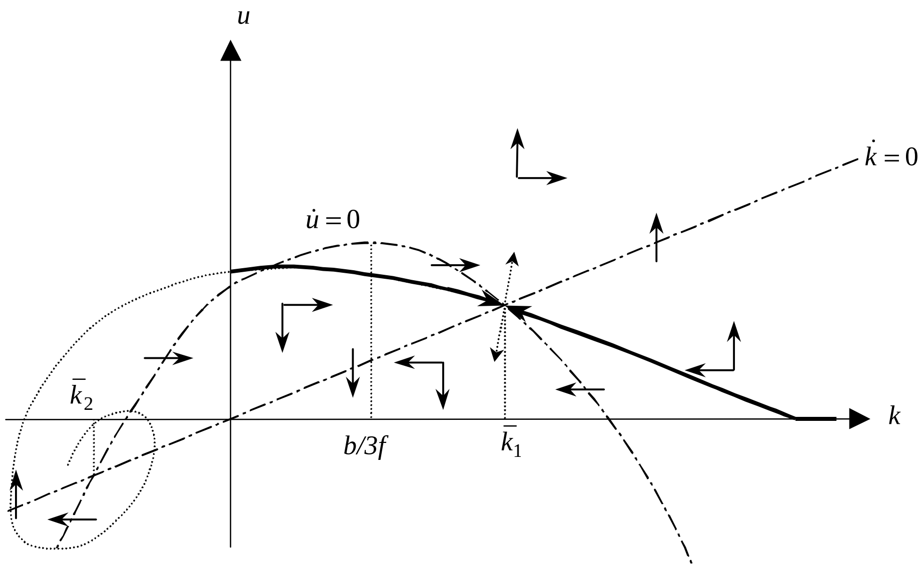

Figure 2: Optimal solution in case of a large first order term in the revenue function, i.e., for $c(\rho+\delta) < a$, and where $\bar{k}_1 > \frac{b}{3f}$:

Proof: $-d(\rho+\delta)^{-1} + b + \sqrt{(b - d(\rho+\delta)^{-1})^2 + 3f(a - (\rho+\delta)c)} < b + \sqrt{b^2 + 3af} \iff$
$\sqrt{(b - d(\rho+\delta)^{-1})^2 + 3f(a - (\rho+\delta)c)} < \sqrt{b^2 + 3af} + d(\rho+\delta)^{-1} \iff$
$(b - d(\rho+\delta)^{-1})^2 + 3f(a - (\rho+\delta)c) < b^2 + 3af + 2d(\rho+\delta)^{-1}\sqrt{b^2 + 3af} + (d(\rho+\delta)^{-1})^2 \iff$
$-2bd(\rho+\delta)^{-1} - 3f(\rho+\delta)c < 2(d(\rho+\delta)^{-1})\sqrt{b^2 + 3af};$
which is always true, since all parameters are positive. $\square$

4 Different Scenarios

Each of the scenarios listed in Proposition 1 has its own qualitative aspects concerning the optimal solution. Below, we study Scenarios 1, 2, and 4 separately. Scenario 3 is most intensively analyzed in Section 5.

4.1 Scenario 1: Large first order term in the revenue function, i.e., $a > c(\rho+\delta)$

From Proposition 1 we know that in this scenario there exists a unique equilibrium in the relevant region ($k > 0; u > 0$; first quadrant), which is a saddle point. Furthermore the second equilibrium is unstable and located in the third quadrant. The corresponding phase diagram is depicted in Figure 2.
From Figure 2 we obtain that the reverse flexible accelerator property obtained by Barucci (1998) does not hold in this scenario, i.e., investment decreases with capital stock in the growth phase. The reason is that the saddle point is located in the concave part of the revenue function. Note that the inflection point of the revenue function

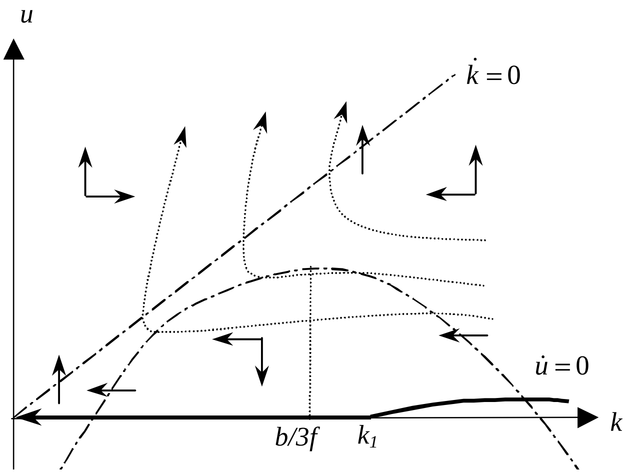

Figure 3: Optimal solution in case of small first order and large third order terms in the revenue function

coincides with the top of the $\underline{u} = 0$-isocline.
The latter implies that when the larger equilibrium $\overline{k}$ has a capital stock value that falls below $b/3f$ the reverse flexible accelerator property will hold in the neighborhood of this equilibrium; see Barucci (1998). This means that investment is lower the larger the difference between the steady state level and the current level of capital goods. The economic intuition behind this is that, as long as the convex part of the revenue function applies, marginal revenue increases with the capital stock so that investment is more profitable if the capital stock is large.
For large capital stocks it can even happen that zero investment is optimal, which can be explained by the fact that the negative third order term in the revenue function is dominant for large values of k.

4.2 Scenario 2: Small first order and large third order terms in the revenue function

From Proposition 1 we know that in this scenario, where the first order term a in the revenue function is small, $a < c(\rho + \mu)$, the third order term is large $f > \dfrac{(b - d(\rho + \mu)\mu)^2}{3((\rho + \mu)c - a)}$ there exists no equilibrium. This is reasonable, since the large f makes high values of the capital stock unattractive, while the small a makes small values of the capital stock unattractive. Thus it does not pay to invest if capital stock is sufficiently small/large. However, since this scenario admits reasonably high values of the second order term b, it is possible that values k_1 and k_2 exist such that investments are positive for $k \in (k_1; k_2)$, as depicted in Figure 3.

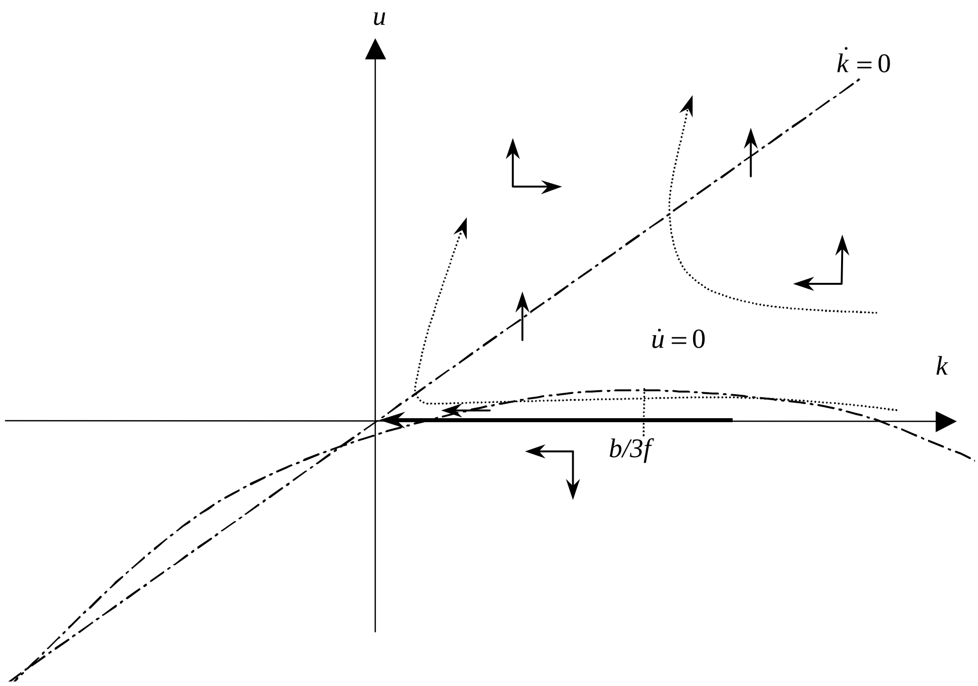

Figure 4: Optimal solution in case of small parameter values in the revenue function

4.3 Scenario 4: Small parameter values in the revenue function

From Proposition 1 we know that when the first order term a in the revenue function is small, $a < c(\rho + \delta)$, the second order term b is small, $b < d(\rho + \delta)\delta$, and also the third order term is small, $f < \dfrac{(b - d(\rho + \delta)\delta)^2}{3((\rho + \delta)c - a)}$, there exists no equilibrium in the first quadrant, since for both equilibria it holds that capital stock is negative. In fact this means that practically the same solution will occur as in Scenario 2. From an economic point of view this is understandable since revenue is low in this scenario; see Figure 4

5 Scenario 3: Small first order and third order terms and large second order term

From Proposition 1 we know that in this scenario where the first order term a in the revenue function is small, $a < c(\rho+\delta)$, the second order term b is large, $b > d(\rho+\delta)\delta$, and the third order term is small, $f < \dfrac{(b - d(\rho + \delta)\delta)^2}{3((\rho + \delta)c - a)}$; there exist two positive equilibria in the relevant region ($k > 0; u > 0$, first quadrant). The large equilibrium is a saddle point while the small equilibrium is unstable.
We first consider the unstable equilibrium to find out under what parameter values it is a focus or a node. Then, we present the phase portraits for both cases.

Remark 1 Since it is not known beforehand whether the revenue function is convex or concave in the large equilibrium, we could distinguish among the subcases that $\overline{k}_1 > b/3f$

and $\bar{k}_1 < b/3f$, respectively (see also Scenario 1).

$$\bar{k}_1 = \frac{(b - d(\rho + \delta)\delta) + \sqrt{(b - d(\rho + \delta)\delta)^2 + 3f(a - (\rho + \delta)c)}}{3f} > \frac{b}{3f} \quad (\ )$$

$$(b - d(\rho + \delta)\delta)^2 + 3f(a - (\rho + \delta)c) > (d(\rho + \delta)\delta)^2 \quad (\ )$$

$$b^2 - 2d(\rho + \delta)\delta) + 3f(a - (\rho + \delta)c) > 0 \quad (\ )$$

$$f < \frac{b^2 - 2b(d(\rho + \delta)\delta)}{3((\rho + \delta)c - a)}.$$

5.1 Node or Focus

We start out by rewriting the conditions under which the two steady states exist:

Lemma 3 Consider Scenario 3 of Proposition 1, where $a < c(\rho + \delta)$ and $b > d(\rho + \delta)\delta$. Then the two steady states exist iff

$$b > \delta d(\rho + \delta) + \sqrt{3f((\rho + \delta)c - a)}. \qquad (17)$$

Proof. From the expressions of the steady states it is clear that they do not exist if

$$(\delta d(\rho + \delta) - b)^2 < 3f((\rho + \delta)c - a).$$

Taking the square root, while taking into account that $b > d(\rho + \delta)\delta$, completes the proof. ■

The next lemma establishes the condition under which the unstable steady state is always a node.

Lemma 4 The unstable steady state is a node iff

$$(2\delta + \rho)^2 - 4\frac{bf - y}{df} > \frac{4}{df}\sqrt{y^2 - 3f((\rho + \delta)c - a)}. \qquad (18)$$

where

$$y = b - \delta d(\rho + \delta) \qquad (19)$$

Proof. The Jacobian of the system $k' = u - \delta k$ and $u' = (\rho + \delta)u + \frac{(\rho+\delta)c - a}{2d} - \frac{b}{d}k + \frac{3f}{2d}k^2$ is

$$\begin{pmatrix} -\delta & 1 \\ -\frac{b}{d} + \frac{3f}{d}k & \rho + \delta \end{pmatrix},$$

so that the eigenvalues are

$$\frac{1}{2}\left(\rho \pm \sqrt{(2\delta + \rho)^2 - 4\frac{b - 3fk}{d}}\right).$$

Hence, a node occurs when

$$(2\delta + \rho)^2 - 4\frac{b - 3fk}{d} > 0.$$

In this expression the unstable steady state, given by $k = \frac{1}{3f}(y - \sqrt{x})$, can be inserted, where

$$x = y^2 - 3f\left((\rho + \mu)c - a\right). \tag{20}$$

This leads to

$$(2\mu + \rho)^2 - 4\frac{b - 3\frac{1}{3f}(y - \sqrt{x})}{d} > 0,$$

which implies that

$$(2\mu + \rho)^2 - 4\frac{bf - y}{df} - \frac{4}{df}\sqrt{x} > 0.$$

Using (20), this last inequality can be rewritten into

$$(2\mu + \rho)^2 - 4\frac{bf - y}{df} > \frac{4}{df}\sqrt{y^2 - 3f\left((\rho + \mu)c - a\right)},$$

which completes the proof. ■

Lemma 5 If it holds that

$$(2\mu + \rho)^2 - 4\frac{bf - y}{df} < 0, \tag{21}$$

then the unstable equilibrium is always a focus.
Condition (21) can be rewritten into

$$b < \frac{1}{4}d\left(4\mu(\rho + \mu) + \frac{1}{f - 1}\rho^2 f\right) \quad \text{for } f < 1, \tag{22}$$

and

$$b > \frac{1}{4}d\left(4\mu(\rho + \mu) + \frac{1}{f - 1}\rho^2 f\right) \quad \text{for } f > 1. \tag{23}$$

Proof. It directly follows from Lemma 4 that under (21) the unstable equilibrium is always a focus. The fact that (22) and (23) are equivalent to (21) can be derived as follows. Expression (21) can be rewritten into

$$4\mu(\rho + \mu) + \rho^2 - \frac{4bf - 4(b - \mu d(\rho + \mu))}{df} < 0,$$

which is in turn equal to the following inequality:

$$4b(1 - f) - 4\mu d(\rho + \mu)(1 - f) + df\rho^2 < 0.$$

The last expression directly leads to (22) and (23), so that the proof is completed. ■

Remark 2 Note that for $f < 1$ the right hand side of (22) is less than the right hand side of (17), which means that the unstable steady state is always non-existent in this case.

What remains to be investigated is the complement of (21), thus the scenario where it holds that

$$(2\mu + \rho)^2 - 4\frac{bf - y}{df} > 0.$$

Under this expression it is not clear whether inequality (18) holds or not. To evaluate this expression we square both sides of (18) to obtain:

$$\left((2\mu + \rho)^2 - 4\frac{bf - y}{df}\right)^2 > \left(\frac{4}{df}\right)^2 \left[y^2 - 3f\left((\rho + \mu)c - a\right)\right]. \qquad (24)$$

which is equivalent to

$$\left((2\mu + \rho)^2 - 4\tfrac{bf - y}{df}\right)^2 - \left(\tfrac{4}{df}\right)^2 \left(y^2 - 3f\left((\rho + \mu)c - a\right)\right) =$$
$$16\tfrac{f-2}{d^2 f} b^2 - \tfrac{8}{df}\left(\rho^2 f + 4\mu\rho f - \rho^2 - 8\mu\rho - 8\mu^2 + 4\mu^2 f\right) b + \rho^4 - 40\mu^2\tfrac{\rho^2}{f} - 64\mu^3\tfrac{\rho}{f} - 8\tfrac{\rho^3}{f}\mu +$$
$$32\mu^3\rho + 16\mu^4 + 24\mu^2\rho^2 + 8\mu\rho^3 - 32\tfrac{\mu^4}{f} + \tfrac{48}{d^2 f}c\rho + \tfrac{48}{d^2 f}c\mu - \tfrac{48}{d^2 f}a > 0.$$

Since this inequality directly follows from (18), the unstable steady state is a node when the inequality holds and a focus otherwise. The left hand side of this inequality is a quadratic expression in b, from which the two roots are:

$$b_1 = \frac{d\left(\rho^2(f - 1) + 4\mu(f - 2)(\mu + \rho)\right) - \sqrt{48(f - 2)(a - c(\mu + \rho)) + d^2\rho^4}}{4(f - 2)}$$

and

$$b_2 = \frac{d\left(\rho^2(f - 1) + 4\mu(f - 2)(\mu + \rho)\right) + \sqrt{48(f - 2)(a - c(\mu + \rho)) + d^2\rho^4}}{4(f - 2)}.$$

Based on the quadratic equation in b and the roots b_1 and b_2, the following conclusions can be drawn:

Conclusion 1. b_1 and b_2 are the roots of a concave parabola in case

$$f < 2.$$

Since $c(\rho + \mu) > a$ here (Scenario 3 of Proposition 1), $f < 2$ implies that both roots b_1 and b_2 exist. The implication is that the unstable steady state is a focus for $b < b_2$ or $b > b_1$, and a node for $b_2 < b < b_1$.

Conclusion 2. b_1 and b_2 are the (existing) roots of a convex parabola if

$$2 < f < 2 + \frac{d^2\rho^4}{48(c(\mu + \rho) - a)}.$$

This implies that the unstable steady state is a focus for $b_1 < b < b_2$, and a node for $b < b_1$ or $b > b_2$.

Conclusion 3. No real roots exist if for

$$f > 2 + \frac{d^2\rho^4}{48\,(c\,(\mu+\rho) - a)}.$$

The implication is that always a node occurs here.
The results of Lemmas 3 and 5, Remark 2, and Conclusions 1-3 are collected in the following proposition.

Proposition 6 The properties of the unstable equilibrium under different parameter values are as follows.

Possibility 1: $f < 1$.

1. For $b < \mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)}$ no steady states exist.
2. For $b > \mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)}$ the unstable steady state is a focus if $b < b_2$ or $b > b_1$ and it is a node if $b_2 < b < b_1$, where

$$b_1 = \frac{d\,(\rho^2\,(f-1) + 4\mu\,(f-2)\,(\mu+\rho)) - \sqrt{48\,(f-2)\,(a - c\,(\mu+\rho)) + d^2\rho^4}}{4\,(f-2)};$$

and

$$b_2 = \frac{d\,(\rho^2\,(f-1) + 4\mu\,(f-2)\,(\mu+\rho)) + \sqrt{48\,(f-2)\,(a - c\,(\mu+\rho)) + d^2\rho^4}}{4f-8}.$$

Possibility 2: $f > 1$.

1. For $b < \mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)}$ no steady states exist.
2. For $\mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)} < b < \frac{1}{4}d\left(4\mu\,(\rho+\mu) + \frac{1}{f-1}\rho^2 f\right)$ the following scenarios must be distinguished:

2.1 For $f < 2$ the unstable steady state is a focus if $b < b_2$ or $b > b_1$ and a node if $b_2 < b < b_1$.

2.2 For $2 < f < 2 + \frac{d^2\rho^4}{48(c(\mu+\rho)-a)}$ the unstable steady state is a focus if $b_1 < b < b_2$ and a node if $b < b_1$ or $b > b_2$.

2.3 For $f > 2 + \frac{d^2\rho^4}{48(c(\mu+\rho)-a)}$ the unstable steady state is always a node.

3. For $b > \max\left\{\frac{1}{4}d\left(4\mu\,(\rho+\mu) + \frac{1}{f-1}\rho^2 f\right); \mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)}\right\}$ the unstable steady state is a focus.

5.2 Numerical Example.:

Here, we present in a $b-f$-diagram what the consequences are of the just presented proposition. To do so we fix the parameter values as follows:

$$a = 0; \qquad c = 1; \qquad d = .3; \qquad \mu = .1; \qquad \rho = .1 \tag{25}$$

For these parameter values the non-existence boundary from Lemma 3 equals

$$b = \mu d\,(\rho+\mu) + \sqrt{3f\,((\rho+\mu)\,c - a)} = 0.006 + 0.7746\sqrt{f}.$$

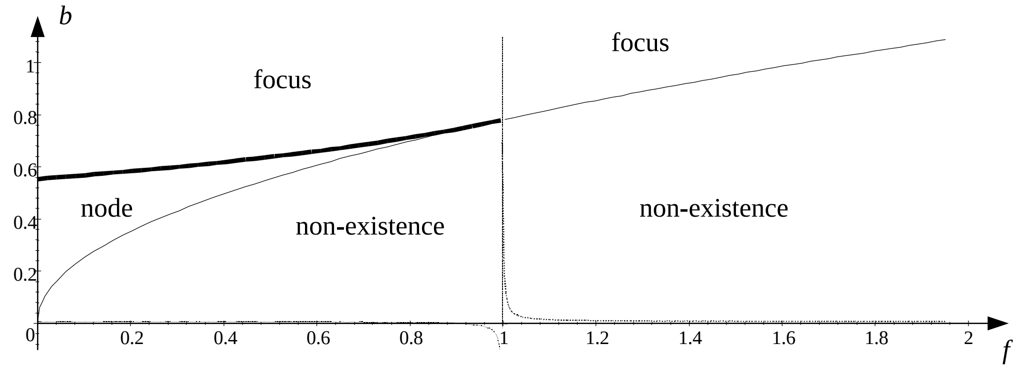

Figure 5: Occurrence of node and focus for different values of b and f. The bold line represents the boundary b_1, the solid line is the non-existence boundary, while the dotted line is the focus boundary.

The focus boundary from Lemma 5 has the following form (note that it is only relevant for $f > 1$):

$$b = \frac{1}{4}d\left(4\mu(\rho+\mu) + \frac{1}{f-1}\rho^2 f\right) = b = 0.006 + 0.00075\frac{f}{f-1}$$

Finally, for b_1 from Proposition 6 we have:

$$\begin{aligned} b_1 &= \frac{1}{4(f-2)}\left(d\left[\rho^2(f-1) + 4\mu(f-2)(\mu+\rho)\right] - \sqrt{48(f-2)(a-c(\mu+\rho)) + d^2\rho^4}\right) \\ &= \frac{1}{8-4f}\left(0.051 - 0.027f + \sqrt{19.2 - 9.6f}\right). \end{aligned}$$

Using these three boundaries, we arrive at Figure 5 which shows for which parameter values of b and f a node or a focus occurs.

Remark 3 Note that in the original Barucci (1998) model $f = 0$. For this model, it can be shown that one unstable equilibrium exists in our Scenario 3. Now we can conclude from Figure 5 that for the Barucci model

- no equilibrium exists for small b (less than $\mu d(\rho+\mu)$)
- the unstable equilibrium is a node for intermediate values of b, and
- the unstable equilibrium is a focus for large values of b.

5.2.1 Sensitivity of the equilibria

It is also interesting to investigate the sensitivity of the equilibria w.r.t. discounting. Given that $b = 0.6$ and $f = 0.1$, we can plot

$$\bar{k}_1 = 1.99 - .1\rho + 3.3333\sqrt{\left[(.597 - .03\rho)^2 - .3\rho - .03\right]}$$

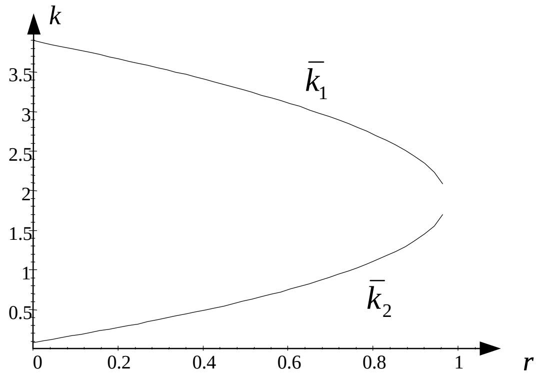

Figure 6: The equilibria as functions of ρ.

$$\bar{k}_2 = 1.99 - .1\rho - 3.3333\sqrt{(.597 - .03\rho)^2 - .3\rho - .03}$$

as a function of the discount rate ρ, which is depicted in Figure 6. In highly myopic cases, i.e. for $\rho > 0.974521$, no equilibrium exists. Also it is not surprising that the capital stock in the stable equilibrium $\bar{k}_1$ decreases with discounting.

5.3 Optimal Investment behavior when the unstable equilibrium is a focus

Since it is not known beforehand whether the revenue function is convex or concave in the large equilibrium, we could distinguish among the subcases that $\bar{k}_1 > b/3f$ and $\bar{k}_1 < b/3f$, respectively (see also Scenario 1 and Remark 1).

5.3.1 Subcase 3A:

If $f < \dfrac{b^2 - ab(d(\rho+\delta)\delta)}{3((\rho+\delta)c - a)}$ then $\bar{k}_1 > b/3f$ is located on the concave part of the revenue function, and the flexible accelerator mechanism keeps on holding in the neighborhood of this equilibrium. This case is depicted in Figure 7.
It is seen in Figure 7 that the stable path converging to $\bar{k}_1$ does not extend to the u-axis. So for small initial values of the capital stock, convergence to the saddle point cannot occur. But there is another solution candidate, namely the solution where $u = 0$ throughout. This is obtained as the solution converging to the equilibrium of the boundary system:

$$u = 0; \quad \dot{k} = -\delta k; \quad \dot{q} = (\rho+\delta)q - r'(k).$$

In the (k, u)-diagram this solution moves to the left on the k-axis and converges to the origin (in infinite time). In the (k, q)-diagram it is a downward sloping saddle point path with the equilibrium $k = 0$ and $q = a/(\rho+\delta)$. This path will intersect the line $q = c$ for some $k = k_{BND}$, and from there on q will be below c and u will be zero because of (8). The exact value of k_{BND} can only be obtained numerically.

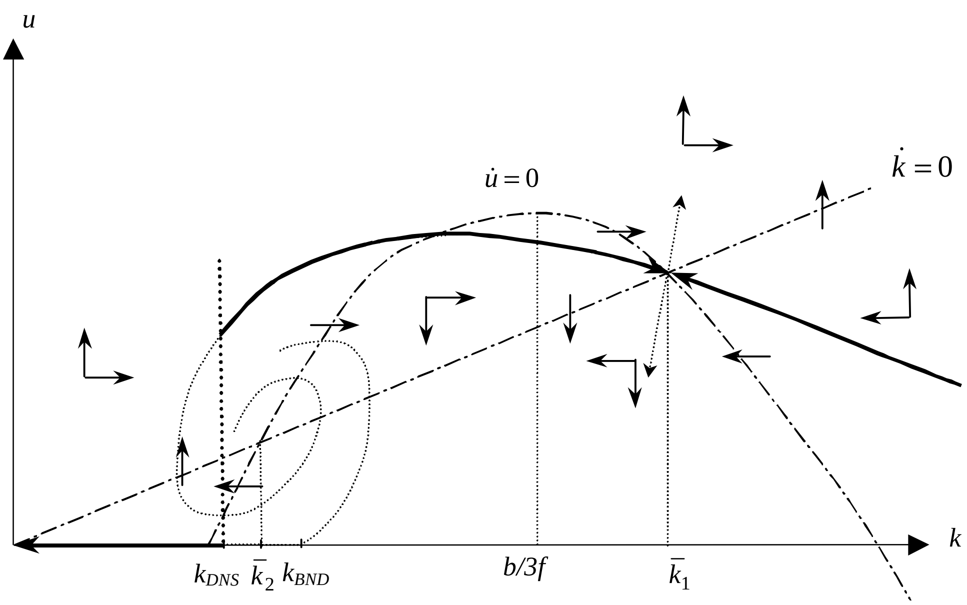

Figure 7: Optimal solution in Subcase 3A with focus and flexible accelerator.

By continuity arguments, it is possible to show that there is a DNS-Point (Dechert-Nishimura-Skiba) - sometimes also called Skiba Point - k_{DNS} so that for $k < k_{DNS}$ it does not pay to reach the larger equilibrium. Rather the left solution candidate with $k \to 0$ is optimal. Economically, this is caused by the fact that the first order term in the revenue function is low so that revenue is low when capital stock is small. This implies that investments are non-profitable. On the other hand, for sufficiently large initial capital stock, $k > k_{DNS}$, it is optimal to choose the saddle point path converging to $\bar{k}_1$: Thus we have identified the two history dependent equilibria $k = 0$ and $k = \bar{k}_1$ here; see also the similar situation on p. 328 in Feichtinger and Hartl (1986).
Note that it is not possible to say which of the values $\bar{k}_2$; k_{BND}, and k_{DNS} is smaller and which is larger.

5.3.2 Subcase 3B:

If $\frac{b^2 - 2b(d(\rho + \mu)\mu)}{3((\rho + \mu)c - a)} < f < \frac{(b - d(\rho + \mu)\mu)^2}{3((\rho + \mu)c - a)}$ then $\bar{k}_1 < b/3f$ is located on the convex part of the revenue function, and the reverse flexible accelerator mechanism holds in the neighborhood of this equilibrium; see Barucci (1998). This means that investment is lower the larger the difference between the steady state level and the current level of capital goods. The economic intuition behind this is that, as long as the convex part of the revenue function applies, marginal revenue increases with the capital stock so that investment is more profitable if the capital stock is large. This is depicted in Figure 8. Except for the slope of the saddle point path near $\bar{k}_1 < b/3f$, the solution is qualitatively the same as in Subcase 3A, including the occurrence of a DNS-point.

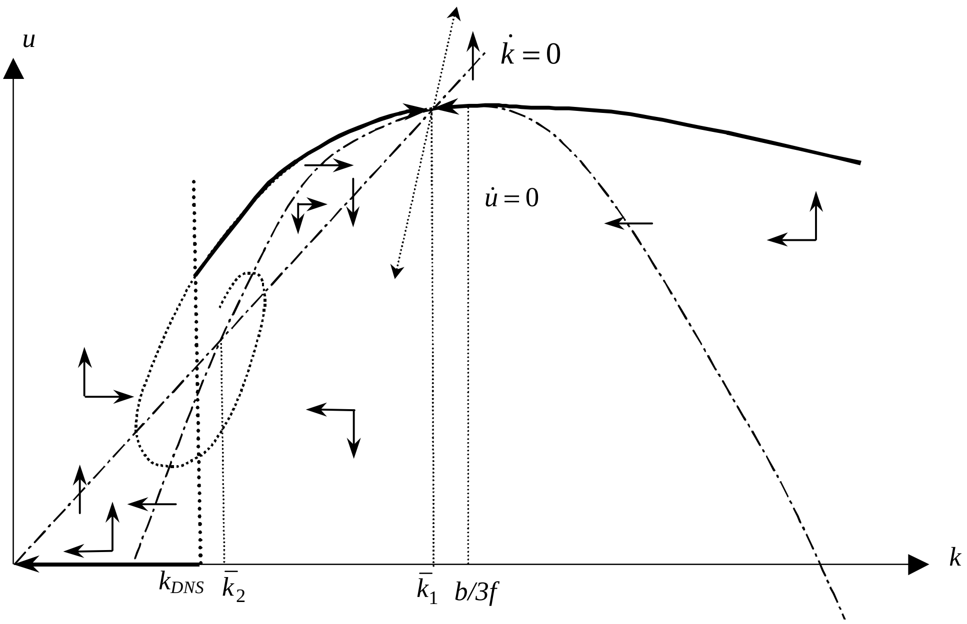

Figure 8: Optimal solution in Subcase 3B with focus and reverse flexible accelerator.

5.4 Optimal Investment behavior when the unstable equilibrium is a node

Since it is again not known beforehand whether the revenue function is convex or concave in the large equilibrium, we could distinguish among the subcases that $\bar{k}_1 > b/3f$ and $\bar{k}_1 < b/3f$, respectively (see also the previous subsection). Here we just consider the case of the flexible accelerator, i.e., $\bar{k}_1 > b/3f$. See Figure 9.
Unlike the case of a focus, here the investment rate is a continuous function of the capital stock. Another difference is that the DNS-point equals the unstable equilibrium itself. Hence the distinction between a focus and a node has its own implications for the firm's investment policy.

Remark 4 In the case of a node, it need not always be the case that the policy function is continuous. In other scenarios it could happen that one-sided or two-sided overlaps of the solution candidates occur.

6 Investment grant

We can also investigate the effect of an investment grant on the optimal firm behavior. Assume that in order to enhance growth or employment the government gives an investment grant of γ per unit of investment (cf. Hartl and Kort 1996): then the investment costs become:

$$c(u) = cu - \gamma u + du^2.$$

As γ gradually increases (from a starting situation with a high c) one passes the following scenarios (here we do not distinguish between focus and node):

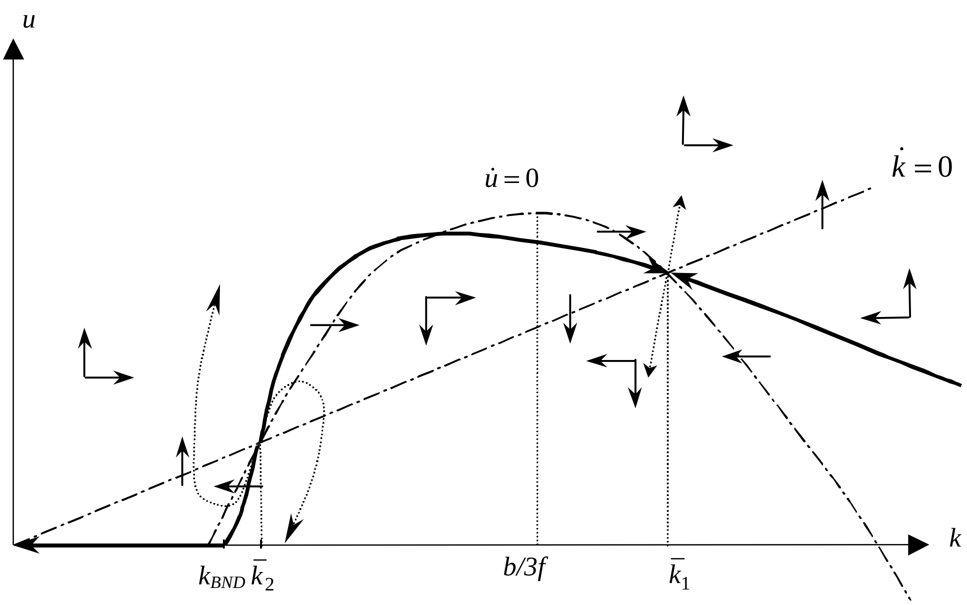

Figure 9: Optimal solution in Subcase 3 with a node and ‡exible accelerator.

² If b > d(½ + ¹)¹: then Scenario 2 ¡ ! Scenario 3 Subcase 3B ¡ ! Scenario 3 Subcase 3A ¡ ! Scenario 1

² If b < d(½ + ¹)¹: then Scenario 2 ¡ ! Scenario 4 ¡ ! Scenario 1

This can be interpreted economically:
For a high investment grant, it is always optimal to reach a steady state capital stock $\bar{k}_1$ (Scenario 1). For small investment grant (in a situation where the investment costs are very high) it is optimal to close the …rm down, u = 0 (Scenarios 2 and 4). If the quadratic part of the revenue function is su¢ciently high, b > d(½ + ¹)¹ then for medium values of the investment grant history dependent equilibria occur. In this case the government can always ensure that the solution candidate with the steady state capital stock $\bar{k}_1$ becomes optimal simply by increasing the investment grant. This is important from a policy maker's point of view. However, on the basis of this analysis it cannot be concluded whether it is in fact optimal for the policy maker to announce such an investment grant. This would require another model where the policy maker is (one of) the decision maker(s).

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