Optimal Pricing of a Product Which Diffuses in Rich and Poor Populations

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Abstract

We consider a market consisting of two populations, termed rich and poor for convenience. If a product is priced such that it is very expensive for the poor, but affordable to the rich, then it becomes a status symbol for the poor and this makes it more desirable for the poor. At a lower price the product is affordable by both populations. However, as more of the poor buy the product, it ceases to be a status symbol, and becomes less appealing to the rich. We present a two-state non-linear optimal control problem that aims to obtain profit-maximizing prices over time in this environment. We find that there are three categories of optimal price paths. One is status symbol pricing with high initial price, declining over time. The other two are mass market pricing, with price declining in one, and increasing and then decreasing in the other.

Key Words

Optimal control, marketing, pricing, market diffusion, aspirational group

1. Introduction

Human societies have a seemingly universal tendency to organize themselves in hierarchies. Members of a society then try to establish ways to be able to assess each other's status in the hierarchical system. Many sociologists have argued that status is evaluated based primarily on material items (e.g.,Goffman 1951). The ability of a product to confer status rests on the extent to which its ownership and consumption is confined to a desirable group which is aspirational to an outside group (Dawson and Cavell 1987). Thus, a high price, by restricting ownership to a small, higher-income group, can create exclusivity and turn the product into a status symbol. This group functions as aspirational for an outside, and relatively larger, lower-income group for which the product becomes exclusive due to its high price. Thus, the product differentiates status between the target group and the outside group. However, since price can rarely guarantee absolute exclusivity, some members of the outside group may be able to acquire the product. Ownership of the exclusive product elevates the status of those persons who manage to acquire it because of its association with membership in the aspirational group. This in turn makes the product more desirable to members of the outside group. However, this exclusivity is based upon a price high enough to make ownership by members of the outside group rare. Otherwise, it is in danger of becoming a "fraudulent symbol" (Goffman 1951), i.e., a symbol that has become diffused across levels of the class hierarchy and potential owners are not assured that consumption and ownership of these products guarantees a certain level of status. Thus, appropriately priced, the brand differentiates status within the outside group. Because members of the high-income group can easily acquire the product, it is not exclusive for them. However, as ownership of the product by the outside group becomes more widespread, the product becomes less desirable to the high-income group. Pantzalis (1995) contains an extensive discussion of exclusivity phenomena, and experimental evidence supporting them. The paragraph above is based on that discussion.

Rao and Sethi (1995) have modeled the situation described above as follows. They consider a monopolist who markets a durable that is purchased only once. The market consists of two populations, called for simplicity rich and poor and subscripted by 1 and 2 respectively. Let $F_i(t)$, $0 \leq F_i(t) \leq 1$, i = 1, 2, denote the proportion of each population which has adopted the brand by time t. An alternative interpretation of $F_i(t)$ is that it is the probability that a member of population i has purchased the product by time t. The corresponding probability density is $dF_i(t)/dt$ Therefore, for population i, $[1/(1 - F_i(t))]\, dF_i(t)/dt$ represents the hazard rate or the conditional probability density that a member of population i makes a purchase at time t given that he has not done so yet.

Let a_i be the price at and beyond which demand from population i is zero, $a_2 < a_1$. The demand for the product in population i is modeled as $a_i - p$, for $0 \leq p \leq a_i$. As adoption in the poor population increases (i.e., F_2 increases), the product becomes less desirable for the rich population. However as adoption in the rich population increases (i.e., F_1 increases), with F_2 held constant, the product establishes itself as a status symbol for the poor population. Using these arguments, Rao and Sethi propose the following model for the diffusion of the product in the two populations:

$$\frac{1}{1-F_1(t)}\frac{dF_1(t)}{dt} = (1-F_2(t))[a_1 - p(t)], \tag{1}$$

$$\frac{1}{1-F_2(t)}\frac{dF_2(t)}{dt} = F_1(t)\max\{0, a_2 - p(t)\}. \tag{2}$$

In (1) and (2), the l.h.s. expressions represent the hazard rates in the rich and poor populations, respectively. In (1) the (1 - F_2) term models the reduction of demand for the product in the rich population, as adoption F_2 in the poor population increases. Similarly, in (2) the F_1 term models the increase in demand for the brand in the poor population as penetration in the rich population increases, serving to make it even more of a status symbol.

This model has some elements similar to the well-known diffusion model proposed by Bass (1969). Bass assumes that there is a *single* homogenous population, and customers buy only once. The probability of purchase by an individual who has not purchased *increases* as more customers adopt the product – a word of mouth effect. In our model, the word of mouth type effect on the poor population is based on the penetration of the rich population, and vice-versa. And, the impact of this word of mouth is asymmetric – depressing the purchase probability for the rich population and increasing it for the poor population. In his original paper, Bass does not include price, however, subsequent extensions consider price as a decision variable (see Mahajan, Muller, and Bass, 1990 for a review of papers in this area).

In this paper, we formulate an optimal control problem faced by a profit maximizing monopolist in a market described by (1) and (2). We use optimal control theory along with numerical computations to develop price paths under different parameter constellations. It would appear that there are three possible categories of policies. First, the monopolist can sell to both rich and poor populations throughout, by maintaining $a_2 \geq p$, thus avoiding any effort at creating exclusivity. Second, he could begin with $p > a_2$, selling to the rich population and establishing the product as a status symbol, and then drop the price below a_2 and sell primarily to the poor population. Third, he could maintain $p > a_2$ throughout, and address only the rich population.

From the perspective of optimal control theory, the model presented here is a two-state non-linear optimal control problem. Moreover, unlike in the prey-predator models (see Clarke, 1979),

the rich-poor dynamics here is asymmetrc. This makes the problem quite interesting as well as complex. Furthermore, the model is quite different from the existing optimal control problems in the marketing literature, which have been surveyed in Sethi (1977), Feichtinger, Hartl and Sethi (1994), Mahajan, Muller and Bass (1993), Rao (1993), and Jørgensen (1986).

The plan of the paper is as follows. In Section 2, we formulate the optimization problem and obtain some preliminary results. Section 3 provides the necessary conditions for optimality using the maximum principle. These conditions give rise to various possible scenarios that may arise. Section 4 analyzes the movements between these scenarios depending on the values of the problem parameters. Explicit solutions are provided in some scenarios. In Section 5, we use the results to obtain computational solutions, and provide their interpretation. Section 6 concludes the paper along with suggestions for future research. The Appendix contains the proofs of the Propositions and Lemmas.

2. Model Formulation and Preliminaries

We specify the notation and develop the optimal control problem under consideration.

F_1(t) ... fraction of the rich population that has bought the product by time t, $t \geq 0$,

F_2(t) ... fraction of the poor population that has bought the product by time t, $t \geq 0$,

a_1 ... prohibitive price for the rich market,

a_2 ... prohibitive price for the poor market, $a_2 < a_1$

c ... constant unit cost, $0 \leq c < a_2$

S ... salvage value parameter, $S \geq 0$

M ... size of the poor population compared to the size of the rich population,

$p(t)$ ... price of the product at time t, $t \geq 0$.

The aim of the monopolist is to optimally set the price $p(t)$ over time so that the expected present value of profits from sales to both populations plus the end-of-the-horizon salvage value based on the captured share of the rich population is maximized. Thus, the objective function can be written as:

$$J = \int_0^T \left[\frac{dF_1}{dt} + M \frac{dF_2}{dt} \right] (p - c) dt + SF_1(T) \tag{3}$$

The last term of (3) merits further discussion. In practice, at some time T the monopolist will introduce a new product to replace the existing one and solve the pricing problem again and this process will be repeated. To account for these future product introductions, we have introduced a salvage value term proportional to the fraction of the rich population which has purchased the current product by time T. If this proportion is high, the new product may be able to position itself as an exclusive product. However, if the proportion is low, it will be impossible for it to draw on its heritage to position itself as exclusive. As S increases, these impacts are magnified.

Thus the optimal control problem faced by the monopolist is to maximize (3) subject to:

$$\frac{dF_1}{dt} = (1 - F_1)(1 - F_2)[a_1 - p], \quad F_1(0) = 0, \tag{4}$$

$$\frac{dF_2}{dt} = F_1(1 - F_2)\max\{0, a_2 - p\}, \quad F_2(0) = 0, \tag{5}$$

the control constraint

$$0 \le p(t) \le a_1, \quad t \ge 0, \tag{6}$$

and the state constraints

$$0 \le F_1(t) \le 1, \quad 0 \le F_2(t) \le 1. \tag{7}$$

Before we apply the maximum principle to solve this problem, we will make a few important observations.

***Proposition* 1:** *Along any optimal solution $p \le a_1$, and the state constraints* (7) *hold for every admissible price trajectory.*

***Proof**:* These follow immediately from (4) and (5).

In view of Proposition 1, we can solve the optimal control problem with the control constraint $p(t) \ge 0$ in place of (6) and without the state constraint (7).

Proposition* 2: An upper limit for the profit is $a_1 + Ma_2 + S$.*

***Proof**:* The maximum possible value of F_1 and of F_2 is 1. Thus, the maximum revenue that can be obtained from the rich population is a_1, the maximum revenue that can be obtained from the poor population is Ma_2, and the maximum salvage value is S. Note that this observation is true for any value of T including $T = \infty$. Thus, a_1 and Ma_2 could be termed the sizes of the rich market and the poor market, respectively. The value S as stated earlier represents the maximum value of the goodwill associated with the sales in the rich market.

In the next proposition, we show that even when $T = \infty$ and $c = 0$, no optimal solution can obtain the objective function value equal to the upper bound $a_1 + Ma_2 + S$. However, it is possible in that case to have a profit that is arbitrarily close to the upper bound.

Proposition 3*: For c = 0 and T = ∞, there exists no optimal solution. Nevertheless, one can obtain a solution which gives a profit arbitrarily close to the upper bound $a_1 + Ma_2 + S$.*

Proof*:* It is easy to see that no pricing strategy will give a profit equal to the upper bound $a_1 + Ma_2 + S$. To obtain a profit close to the upper bound, consider a small $\varepsilon > 0$ such that $a_1 - \varepsilon > a_2$ and $a_2 - \varepsilon > 0$. Use the price $p(t) = a_1 - \varepsilon$, $0 \le t \le \tau$, where τ *is* arbitrarily large. Set $p(t) = a_2 - \varepsilon$ for $t \ge \tau$. The profit associated with this policy has the lower bound

$$(a_1 - \varepsilon + S)F_1(\tau) + M(a_2 - \varepsilon)[\mathrm{F}_2(\mathrm{t}) - \mathrm{F}_2(\tau)],$$

where $\mathrm{F}_1(\tau) = 1 - e^{-\varepsilon\tau}$ and $F_2(t) - F_2(\tau) \to 1$ as $t \to \infty$. Note that the profit in the rich market after τ is ignored in the above expression. For ε sufficiently small and τ sufficiently large, this lower bound expression can be made arbitrarily close to the upper bound $a_1 + Ma_2 + S$. This clearly means that there does not exist an optimal solution to the optimization problem under consideration.

Note that when $T = \infty$, the most reasonable value for S is zero since there is only one product to be considered. So for $T = \infty$, the upper bound when $S = 0$ is $a_1 + Ma_2$. This provides a reason for having a finite product life cycle for the product. With $S > 0$, it is then possible to obtain a profit greater than $a_1 + Ma_2$ even when $T < \infty$. From now on, therefore, we will only consider the case $T < \infty$.

3. Necessary optimality conditions

In order to obtain necessary conditions for optimal solution of our control problem, we make use of Pontryagin's Maximum Principle; see Sethi and Thompson (2000) or Feichtinger and Hartl (1986). The Hamiltonian is given by

$$H = (p - c + \lambda_1)(1 - F_1)(1 - F_2)[a_1 - p] + [M(p - c) + \lambda_2]F_1(1 - F_2)\max\{0, a_2 - p\} \qquad (8)$$

where λ_1 and λ_2 denote the co-states or the adjoint variables associated with F_1 and F_2, respectively. They follow the adjoint equations

$$\frac{d\lambda_1}{dt} = (p - c + \lambda_1)(1 - F_2)[a_1 - p] - [M(p - c) + \lambda_2](1 - F_2)\max\{0, a_2 - p\},\ \lambda_1(T) = S, \qquad (9)$$

$$\frac{d\lambda_2}{dt} = (p - c + \lambda_1)(1 - F_1)[a_1 - p] + [M(p - c) + \lambda_2]F_1\max\{0, a_2 - p\},\ \lambda_2(T) = 0. \qquad (10)$$

The interpretation of the adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ are as follows. $\lambda_i(t)$ represents the marginal value of an increase in $F_i(t)$, $i = 1, 2$, expressed in current-value or time-t dollars. Thus, $\lambda_i(t)\,\varepsilon$ represents the increase in the objective function value if $F_i(t)$ increases by a small amount ε

> 0. Since an increase in $F_2(t)$ at any time $0 \le t < T$ hurts the future sales of the product in both rich and poor markets, we should have $\lambda_2(t) < 0$ for all t except at the terminal time T where $\lambda_2(T) = 0$ as required by (10). Indeed, we shall prove this intuition in Proposition 11. On the other hand, an increase in the value of $F_1(t)$ decreases future sales in the rich market and increases future sales in the poor market. Thus, the value of $\lambda_1(t)$ can be positive or negative depending on the parameters of the problem.

Economic interpretations to the adjoint equations (9) and (10) and Hamiltonian in (8) are standard, see Sethi and Thompson [6a, Section 2.2.4].

The Hamiltonian maximization conditions on the feasible price range $[0,a_1]$ is

$$\left.\frac{\partial H}{\partial p}\right|_{p=0} = (1-F_1)(1-F_2)[a_1] + (c-\lambda_1)(1-F_1)(1-F_2) + MF_1(1-F_2)a_2 + [cM-\lambda_2]F_1(1-F_2) \le 0, \quad for\ p=0. \tag{11}$$

$$\frac{\partial H}{\partial p} = (1-F_1)(1-F_2)[a_1-p] - (p-c+\lambda_1)(1-F_1)(1-F_2) + MF_1(1-F_2)[a_2-p] - [M(p-c)+\lambda_2]F_1(1-F_2) = 0, \quad for\ 0<p<a_2 \tag{12}$$

and

$$\frac{\partial H}{\partial p} = (1-F_1)(1-F_2)[a_1-p] - (p-c+\lambda_1)(1-F_1)(1-F_2) = 0, \qquad \text{for } a_2 < p < a_1. \tag{13}$$

The problem with $c > 0$ is not significantly different from the one with $c = 0$, since the former can be reduced to the latter by defining the price control $q = p - c$ and by redefining $a_1 - c$ and $a_2 - c$ as new a_1 and a_2. The control constraint $0 \le p$ reduces to $-c \le q$. Henceforth, we only deal with the case $c = 0$.

3.1. **Analysis in the case of $c = 0$**

For $c = 0$, the equations (9) and (10) reduce to

$$\frac{d\lambda_1}{dt} = (p+\lambda_1)(1-F_2)[a_1-p] - [Mp+\lambda_2](1-F_2)\max\{0, a_2-p\}, \ \lambda_1(T) = S \tag{14}$$

$$\frac{d\lambda_2}{dt} = (p+\lambda_1)(1-F_1)[a_1-p] + [Mp+\lambda_2]F_1 \max\{0, a_2-p\}, \ \lambda_2(T) = 0. \tag{15}$$

Hamiltonian maximization conditions (11)-(13) reduce to

$$\left.\frac{\partial H}{\partial p}\right|_{p=0} = (1-F_2)\{(1-F_1)[a_1-\lambda_1]+F_1[Ma_2-\lambda_2]\} \le 0, \quad \text{for } p=0 \tag{16}$$

$$\frac{\partial H}{\partial p} = (1-F_2)\{(1-F_1)[a_1-2p-\lambda_1]+F_1[Ma_2-2Mp-\lambda_2]\} = 0, \quad \text{for } 0<p<a_2, \tag{17}$$

and

$$\frac{\partial H}{\partial p} = (1-F_1)(1-F_2)[a_1-2p-\lambda_1] = 0, \quad \text{for } p>a_2. \tag{18}$$

From (16), (17) and (18), we can derive the locally maximizing price, respectively, as follows:

$$p = 0, \qquad \text{if } \lambda_1 \ge a_1 + \frac{(Ma_2-\lambda_2)F_1}{1-F_1}, \tag{19}$$

$$p = \frac{(1-F_1)[a_1-\lambda_1]+F_1[Ma_2-\lambda_2]}{2(1-F_1)+2MF_1}, \quad \text{if } a_1 + \frac{(Ma_2-\lambda_2)F_1}{1-F_1} > \lambda_1 > a_1 - 2a_2 - \frac{(Ma_2+\lambda_2)F_1}{1-F_1}. \tag{20}$$

Note that the upper limit (l.h.s.) in (20) is always bigger than the lower limit (r.h.s.) so that the specified interval for λ_1 is not empty. Note also that

$$p = \frac{a_1-\lambda_1}{2}, \text{ if } -a_1 \le \lambda_1 < a_1 - 2a_2. \tag{21}$$

***Lemma* 1:** *For $p > a_2$ it holds that $\lambda_1 \ge -a_1$.*

Proof: By Proposition 1, $p < a_1$ which, by (21) implies the result.

Note that the conditions on λ_1 in (20) and (21) are only necessary conditions because there may be an overlap of the corresponding regions. This will be investigated further in the next section.

3.2. Further analysis of the Hamiltonian maximization condition

First we note that the Hamiltonian has a kink w.r.t. p at point a_2. The jump in the derivative is

$$\left.\frac{\partial H}{\partial p}\right|_{p=a_2^+} - \left.\frac{\partial H}{\partial p}\right|_{p=a_2^-} = [Ma_2+\lambda_2]F_1(1-F_2). \tag{22}$$

Clearly, this expression can be positive or negative. Thus, we have to consider two different regions depending on the sign of $Ma_2+\lambda_2$.

3.2.1. Region 1: $Ma_2 + \lambda_2 > 0$

In this region the Hamiltonian has, by (22), a convex kink w.r.t. p and there may be two local maxima. The result can be summarized as follows:

***Proposition* 4:** *Let* $Ma_2 + \lambda_2 > 0$, *and define*

$$\lambda^* = a_1 - 2a_2 - \left(a_2 + \frac{\lambda_2}{M}\right)\left[\sqrt{1 + \frac{MF_1}{1-F_1}} - 1\right]. \tag{23}$$

(a) If $\lambda_1 \geq a_1 + \dfrac{(Ma_2 - \lambda_2)F_1}{1-F_1}$, *then* $p = 0$ *is optimal.*

(b) If $a_1 + \dfrac{(Ma_2 - \lambda_2)F_1}{1-F_1} > \lambda_1 > a_1 - 2a_2$, *then* $p = \dfrac{(1-F_1)[a_1 - \lambda_1] + F_1(Ma_2 - \lambda_2)}{2(1-F_1) + 2MF_1}$ *is a local maximum.*

(c) If $a_1 - 2a_2 > \lambda_1 > a_1 - 2a_2 - \dfrac{(Ma_2 + \lambda_2)F_1}{1-F_1}$, *then H has two local maxima. Evaluating H at both of these maxima and identifying the global maximum yields the following three subcases:*

(c_1) *If* $a_1 - 2a_2 \geq \lambda_1 > \lambda^*$, *then* $p = \dfrac{(1-F_1)[a_1 - \lambda_1] + F_1(Ma_2 - \lambda_2)}{2(1-F_1) + 2MF_1} < a_2$ *is optimal.*

(c_2) *If* $\lambda_1 = \lambda^*$, *then* $p = \dfrac{(1-F_1)[a_1 - \lambda_1] + F_1(Ma_2 - \lambda_2)}{2(1-F_1) + 2MF_1}$ *and* $p = \dfrac{a_1 - \lambda_1}{2}$ *are both optimal.*

(c_3) *If* $\lambda^* > \lambda_1 \geq a_1 - 2a_2 - \dfrac{(Ma_2 + \lambda_2)F_1}{1-F_1}$, *then* $p = \dfrac{a_1 - \lambda_1}{2} > a_2$ *is optimal.*

(d) If $-a_1 \leq \lambda_1 < a_1 - 2a_2 - \dfrac{(Ma_2 + \lambda_2)F_1}{1-F_1}$, *then* $p = \dfrac{a_1 - \lambda_1}{2}$ *is optimal.*

***Proof**:* See Appendix.

Figure 1 shows the value of the expression $H/(1-F_2)$ as a function of p in each of the above cases. The price at which the maximum of this expression occurs is indicated by solid-lined circles. In case (a) the maximum occurs at 0. In cases (b) and (c_1) it occurs between 0 and a_2. In case (c_2) there are two maxima, one between 0 and a_2 and one between a_2 and a_1. Finally in cases (c_3) and (d) the maximum occurs between a_2 and a_1. Note that $H(a_1) = 0$ from (8) and that we have depicted an instance of $H(0) > 0$ in Figure 1. In Figure 2 the maximizing price in each of the cases is shown as a function of λ_1. Note that the case (c_2), where there are two maximizing prices, occurs at $\lambda_1 = \lambda^*$. This implies a downward jump in price at any time t when the λ_1 trajectory crosses the value λ^*; see the example in Figure 8.

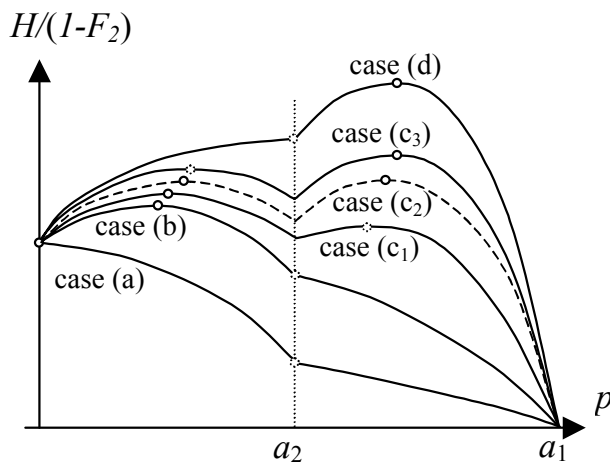

Figure 1. The different cases in Region 1: in the hairline case (c_2) the two local maxima in the two regions are both globally optimal

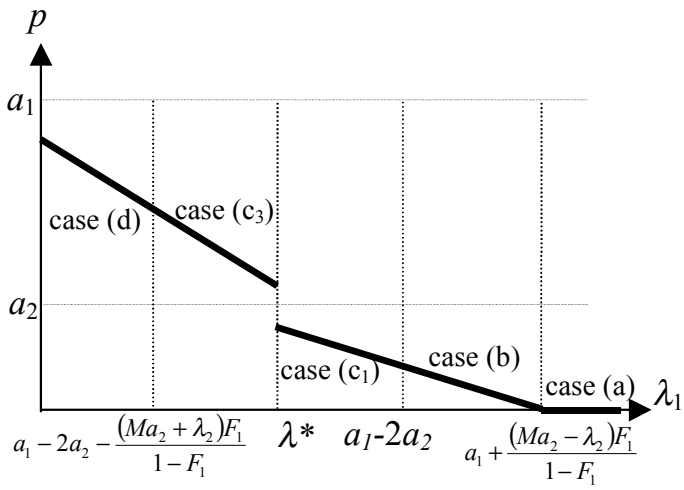

Figure 2. The optimal price $p(t)$ expressed in terms of λ_1 for a fixed λ_2. Note that the line $p = \frac{a_1 - \lambda_1}{2}$ in cases (d) and (c_3) is steeper than the line $p = \frac{(1-F_1)[a_1-\lambda_1] + F_1(Ma_2-\lambda_2)}{2(1-F_1)+2MF_1}$ in cases (b) and (c_1)

3.2.2. Region 2: $Ma_2 + \lambda_2 \le 0$

In this region the Hamiltonian has, by (22), a concave kink (if $Ma_2 + \lambda_2 < 0$) w.r.t. p and therefore Hamiltonian is concave in the control variable p. The result in this case is summarized in the following proposition:

Proposition 5: *Let* $Ma_2 + \lambda_2 \le 0$. *In every case there is only one local max that is also optimal:*

(a) If $\lambda_1 \ge a_1 + \frac{(Ma_2-\lambda_2)F_1}{1-F_1}$, *then* $p = 0$ *is optimal.*

(b) If $a_1 + \frac{(Ma_2-\lambda_2)F_1}{1-F_1} > \lambda_1 > a_1 - 2a_2 - \frac{(Ma_2+\lambda_2)F_1}{1-F_1}$, *then* $p = \frac{(1-F_1)[a_1-\lambda_1] + F_1(Ma_2-\lambda_2)}{2(1-F_1)+2MF_1}$ *is optimal.*

(d) If $\lambda_1 < a_1 - 2a_2$, *then* $p = \frac{a_1-\lambda_1}{2}$ *is optimal.*

(e) If $a_1 - 2a_2 - \frac{(Ma_2+\lambda_2)F_1}{1-F_1} > \lambda_1 > a_1 - 2a_2$, *then* $p = a_2$ *is optimal.*

Proof: See Appendix.

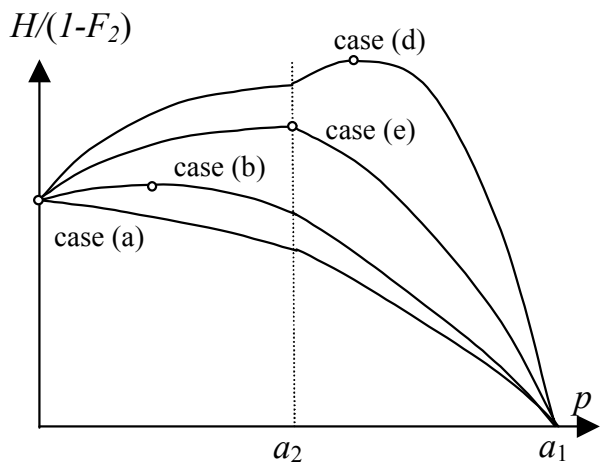

Figure 3. The different cases in Region 2

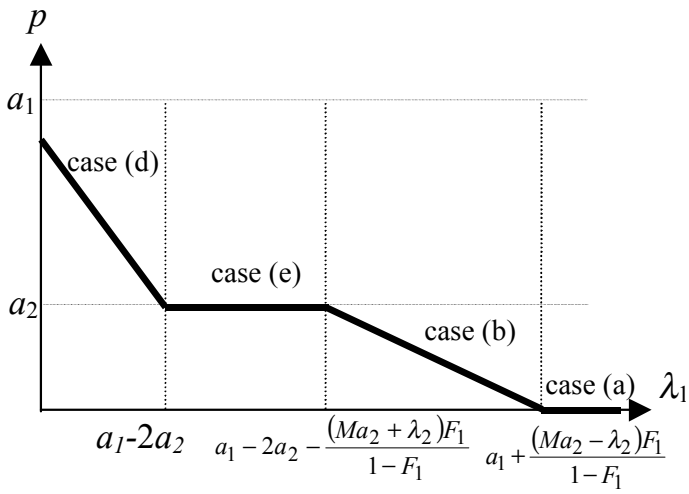

Figure 4. The optimal price $p(t)$ expressed in terms of λ_1 for a fixed λ_2. Note that the line $p = \frac{a_1 - \lambda_1}{2}$ in case (d) is steeper than the line $p = \frac{(1-F_1)[a_1 - \lambda_1] + F_1(Ma_2 - \lambda_2)}{2(1-F_1) + 2MF_1}$ in case (b)

These cases are depicted in Figure 3. Once again we have depicted an instance of $H(0) > 0$ in Figure 3. In Figure 4 the price is shown as a function of λ_1. It is interesting to contrast these figures with Figures 1 and 2. In cases (c_1), (c_2), and (c_3) in Figures 1 and 2, we have two locally maximizing prices, one above and one below a_2. In case (e) in Figures 3 and 4, we have a single maximizing price which occurs at a_2 . This results in a price equal to a_2 over an interval of time during which λ_1 takes values in the interval from $a_1 - 2a_2 - \frac{(Ma_2 + \lambda_2)F_1}{1 - F_1}$ to $a_1 - 2a_2$; see the example in Figure 11.

For any given (F_1, F_2), Figure 5 shows the regions in the (λ_1, λ_2) space where the various cases occur. It may be noted that the boundaries separating these regions are straight lines. Note that $a_1 - 2a_2$ can be negative, which in turn implies that $S^* = a_1 - a_2\left[1 + \sqrt{1 + \frac{MF_1}{1 - F_1}}\right]$ can be negative. This is reasonable, since if the poor people are willing to pay more than half of what the rich people are willing to pay (i.e., $a_1 < 2a_2$), it does not make much sense to sell only to the rich people. Thus for $a_1 - 2\,a_2 < 0$, the regions where cases (c) and (d) are optimal only occur if $\lambda_1 < 0$.

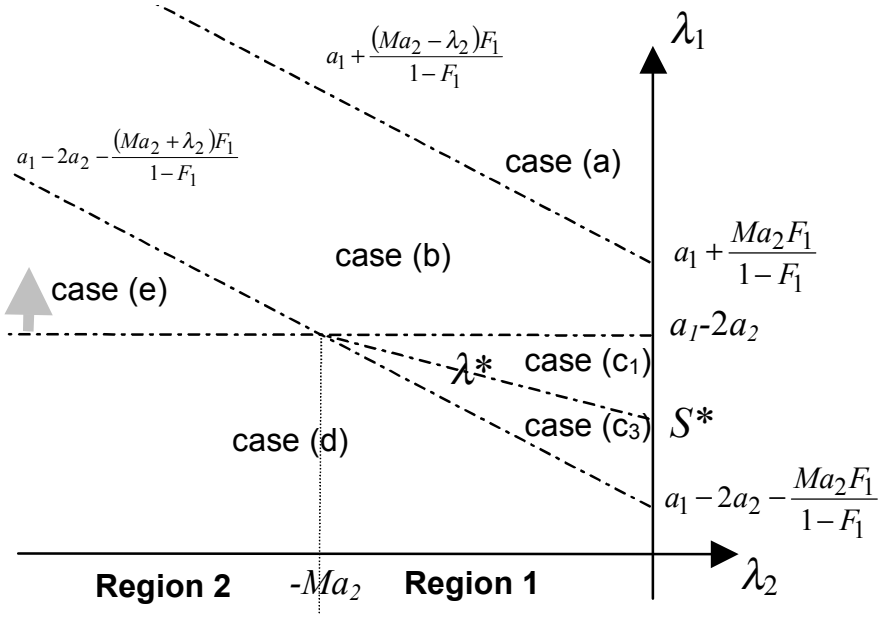

Figure 5. The boundary lines separating the cases in the (λ_1, λ_2) space for any given (F_1, F_2).

In the rest of the paper we will investigate possible movements between these regions, and provide some typical time paths of price and market shares.

4. Analysis of the movement between and within the regions

In the previous section we have investigated in detail which price is optimal for which combination of the shadow prices (λ_1, λ_2) and have identified various cases shown in Figure 5. For the purposes of the following analysis, it is convenient to group these cases into four possible scenarios on the basis of the price ranges that apply in these cases.

Scenario 1: $p = 0$, i.e., Case (a)

Scenario 2: $0 < p < a_2$, i.e., Cases (b) and (c_1)

Scenario 3: $p = a_2$, i.e., Case (e)

Scenario 4: $a_2 < p < a_1$, i.e., Cases (c_3) and (d)

Note that we have not included (c_2) in any of the scenarios. This is because case (c_2) has only a fleeting existence at an instant of time when the (λ_1, λ_2) trajectory crosses the boundary marked λ^*.

In what follows we first state the results concerning the movement between and within the scenarios/cases. Proofs of these results are in Appendix. Then we discuss the implications of these results.

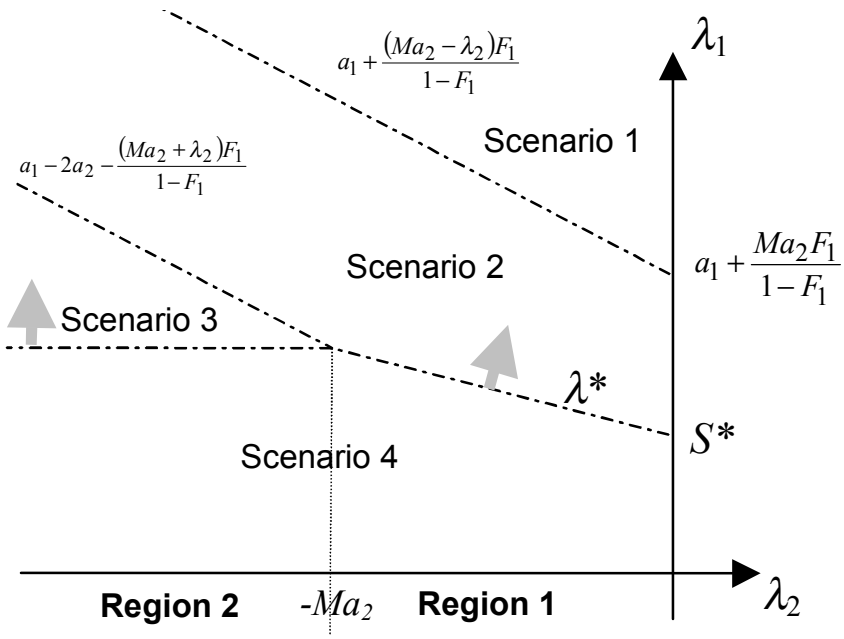

Figure 6. The boundary lines separating the scenarios in the (λ_1, λ_2) space for any given (F_1, F_2).

Proposition 6: *If the optimal solution stays in Scenario* 1 *from any time* t_1 *on, then the penetration of the rich market has the following upper bound:*

$$F_1(t) \le 2 - B(F_2(t)) + \sqrt{B(F_2(t))^2 - 2B(F_2(t)) - 1}\,, \ \forall t \in [t_1, T], \tag{24}$$

where

$$B(F_2(t)) = \frac{\exp[1 - F_1(t_1) + (a_1/a_2)(F_2(t) - F_2(t_1))]}{1 - F_1(t_1)}.$$

Proposition 7:

(i) *A transition from case* (*e*) *to case* (*d*) *is not optimal.*

(ii) *A transition from case* (c_1) *to* (c_3) *is not optimal.*

Proposition 8: *A necessary condition for Scenario* 4 *to hold at the terminal time T is* $a_1 \ge 2a_2$.

Proposition 9: *If Scenario* 4 *holds throughout, then the solution is given by*

$$\lambda_1(t) = -a_1 - \frac{4}{t-k} \quad \textit{where} \quad k = T + \frac{4}{S + a_1},$$

$$p(t) = \frac{a_1 - \lambda_1(t)}{2} = a_1 + \frac{2}{t-k}, \quad (\textit{see Figure } 7)$$

$$F_1(t) = 1 - \left(\frac{(S + a_1)(T - t) + 4}{(S + a_1)T + 4} \right)^2, \textit{ and}$$

$$\lambda_2(t) = \frac{4(S+a_1)^2}{((S+a_1)T+4)^2}(t-T).$$

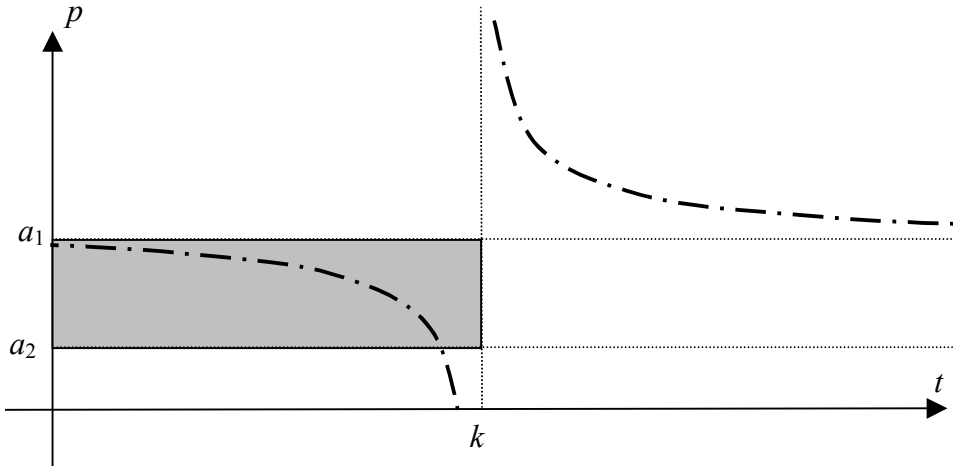

***Figure* 7.** The price trajectory in Scenario 4.

Proposition 10: *If Scenario* 4 *holds at the end of the horizon, then it holds throughout.*

Proposition 11: *The adjoint variable* $\lambda_2(t) < 0 \quad \forall t \in [0,T)$ *and* $\lambda_2(T) = 0$.

Discussion of results

Proposition 6 provides an upper bound on the penetration of the rich market in terms of the penetration of the poor market when $p = 0$ throughout. This suggests that setting $p = 0$ throughout is not optimal and the optimal solution will leave Scenario 1 at some point in time. The resulting increase in price $p = 0$ to a strictly positive value results in a reduced growth of F_2 but the effect on the growth of F_1 is ambiguous. The reasons for the ambiguity are two opposing effects. The direct effect reduces the growth of F_1. Indirectly, the slower growing F_2 yields an increasing growth rate of F_1. If this indirect effect surpasses the direct effect, $F_1(t)$ may then exceed the upper bound given in Proposition 6. Numerical computations confirm this observation.

Proposition 8 says that if a_1 is not sufficiently high compared to a_2, then we will never have an exclusive policy throughout the horizon. This means we can go from Scenario 4 to Scenario 2 but not vice versa.

Proposition 9 gives the price trajectory when the product is sold only to the rich market. This price is decreasing as shown in Figure 7. It may switch to case (c_1), where price is below a_2. Depending on the value of S, this may be the last regime, or it may switch to case (c_1).

5. Numerical Exploration of Policies

We have so far explored the canonical system numerically to try to characterize the types of marketing policies obtained, and their related price paths. We also examined a large number of cases varying each parameter over a wide range, as shown in the Table below:

Parameter	**Values**
a_1	2.5, 5, 10
S	0, 10, 20, 30, 83, 100
M	5, 10, 20
t	0.7, 0.5

Table 1. The parameter values investigated in the numerical analysis

The price to the poor population, parameter a_2, was held at 1 throughout; thus we explored cases where the ratio a_1/a_2 between the reservation prices for the rich and poor populations was 2.5, 5 and 10. Since a_1 and Ma_2 are the sales potentials of the rich and poor markets respectively, our parameter choices enabled us to examine poor markets that are from 0.5 to 8 times the size of the rich market.

Three broad categories of policies emerge. The first category consists of price paths which we will refer to as exclusivity strategies. This category of policies emerges when the poor market is not more than twice the size of the rich market and S is small. Here, the price path is a declining one, but with $p > a_2$ for at least the initial period. Then price jumps down to a level below a_2, and declines for the rest of the time horizon. Included in this category are price paths with $p > a_2$ for the entire time horizon. Sales are therefore limited to the rich population for at least the first part of the time horizon, and this makes the product more exclusive for the poor population when it finally becomes affordable to this group towards the end of the time horizon. And, once the price drops, the remaining potential of the rich market is still accessible to the product. This type of strategy may be followed by a monopolist who periodically replaces an existing model of the product with a new one. At the end of the life cycle of each model, the price is dropped, to obtain some additional sales from the poor population without damaging the exclusive image of the product much and still enjoy close to the maximum salvage value. Included in this category of policies is the special case where the price declines to a_2, remains at a_2 for some time (still denying access to the poor population) and then declines below a_2.

Now we relate these price paths to the scenarios and cases discussed earlier. The solutions start in scenario 4 - cases (d) and (c_3) – with $p > a_2$; price then declines until it encounters the border between cases (c_3) and (c_1). There the price jumps and enters scenario 2 – cases (c_3) and (b).

In the special case mentioned, the price path again starts in case (d), enters case (e), and eventually enters case (b). These price paths are illustrated in Figures 8 and 9.

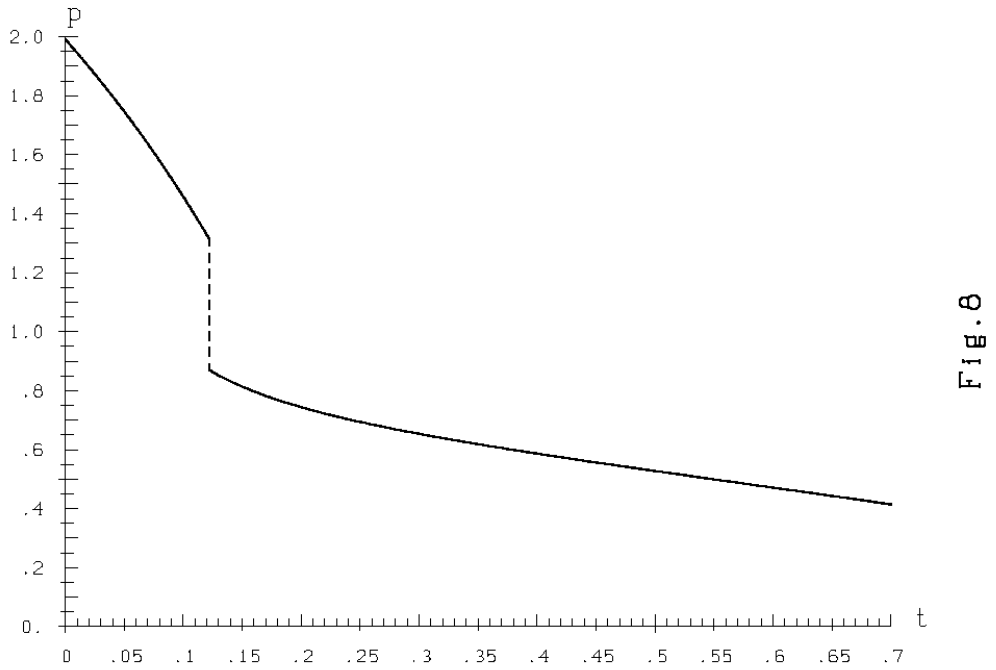

Figure 8. A price path with a jump. Cases (d) - (c_3) - (c_1) - (b). The parameters chosen are $a_1 = 5$, $a_2 = 1$, $S = 30$, $M = 10$, $T = 0.7$.

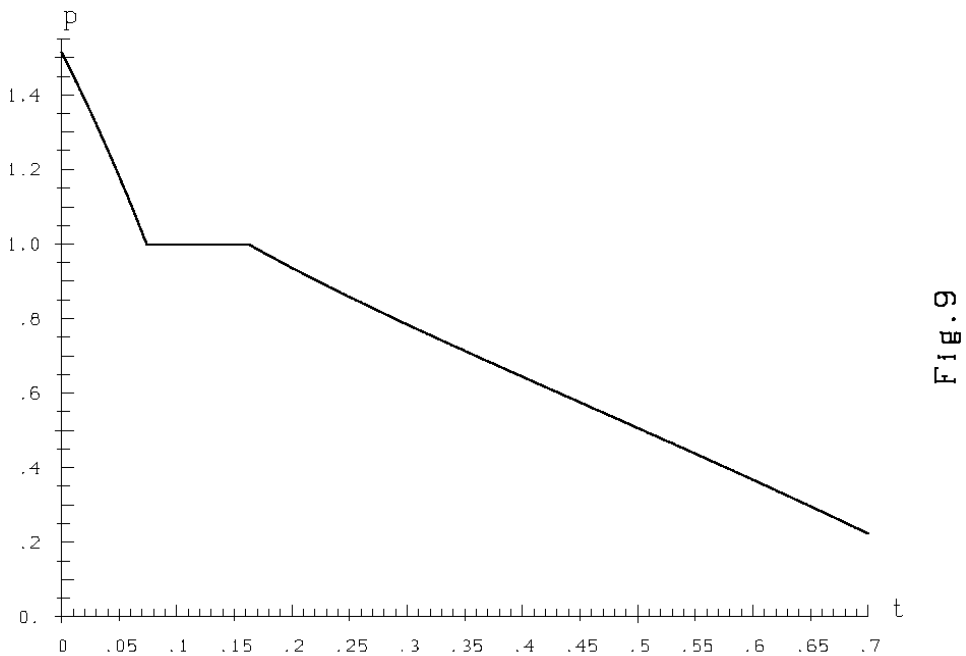

Figure 9. A price path without a jump with constant segment $p = a_2$. Cases (d) - (e) - (b). The parameters chosen are $a_1 = 5$, $a_2 = 1$, $S = 50$, $M = 5$, $T = 0.7$.

The impact of S on the price path is quite significant. When S is very large, the monopolist might do well to essentially give away the product so that $F_1(T)$ is very close to 1, and $SF_1(T)$ is as large as possible. However, when S is small the monopolist makes money by selling the product at a high price, rather than relying on a high salvage value. Thus for the same market potential, the length of time t^* for which $p > a_2$ depends on S – the higher the salvage value S, the lower is the value of t^*.

The second category of policies is observed when the potential of the poor market is large compared to the rich market. Here, $p \leq a_2$ and is monotonically declining throughout. Thus no effort is made to position the product as exclusive. Of course, some members of the rich population purchase the product, but they are doing so because of its price and in spite of the fact that many members of the poor population have purchased the product as well. This price path corresponds to case (b) or (c_1). A typical example is illustrated in Figure 10.

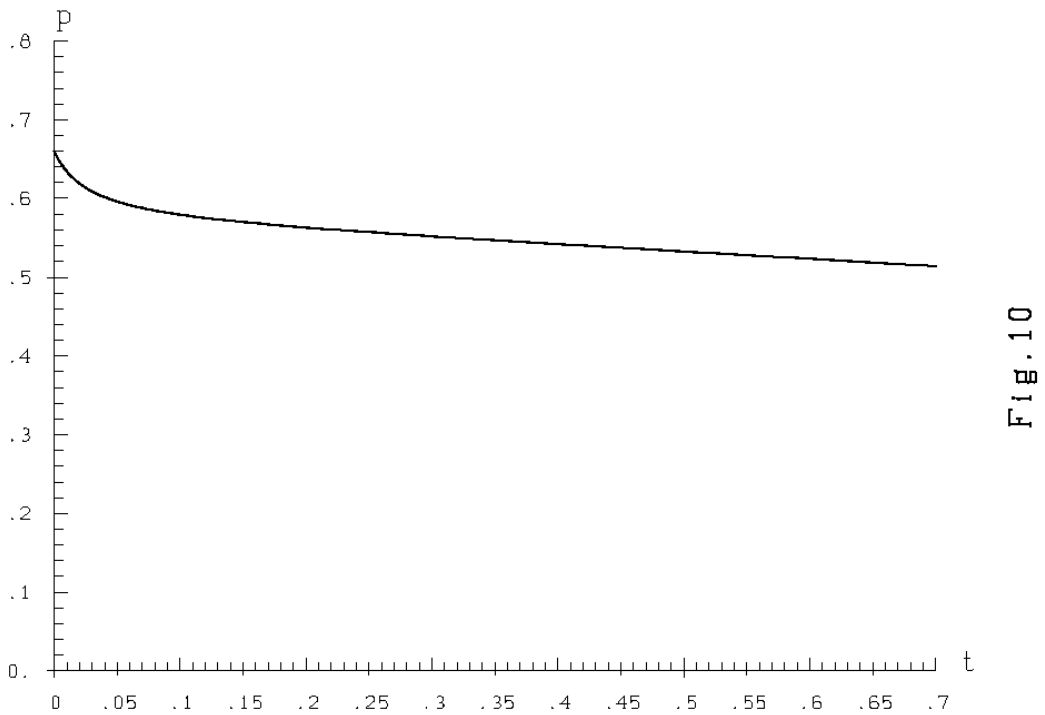

***Figure* 10.** A decreasing price path with $p < a_2$. Cases (b) - (c_1). The parameters chosen are $a_1 = 2.5$, $a_2 = 1$, $S = 0$, $M = 20$, $T = 0.7$.

The third category of policies is also observed for large potential poor markets, but here, we do not have a monotonically decreasing price path, but rather price increases and then declines.

Here $p \leq a_2$ always, so again there is no effort to position the product as exclusive. A special case of this category of price paths occurs when $p = 0$ for a small initial period. In practice, this implies that the product would be given away (or sold at cost) to create a market "buzz", to stimulate sales to both rich and poor populations. One possibility here would be to restrict these giveaways to members of the rich population with attendant publicity so that some exclusivity effect can be obtained, and used to stimulate sales further to the poor population. A second special case ends the time horizon with a brief period with $p = 0$, driving up the salvage value for cases where S is large. This case corresponds to the situation where inventories of an old model of the product are being disposed off in order to clear space for a new model. A third special case arises when the initial increasing price hits $p = a_2$ and stays there a while before it starts declining. The declining phase could terminate with $p < a_2$ or go all the way to $p = 0$. In terms of the scenarios analyzed earlier, we have case (b), with special cases of the price path transiting from case (a) to case (b), from case (a) to case (b) back to case (a) and from case (b) to case (e) and then back to case (b) and then possibly to case (a). Figures 11-14 illustrate these price paths.

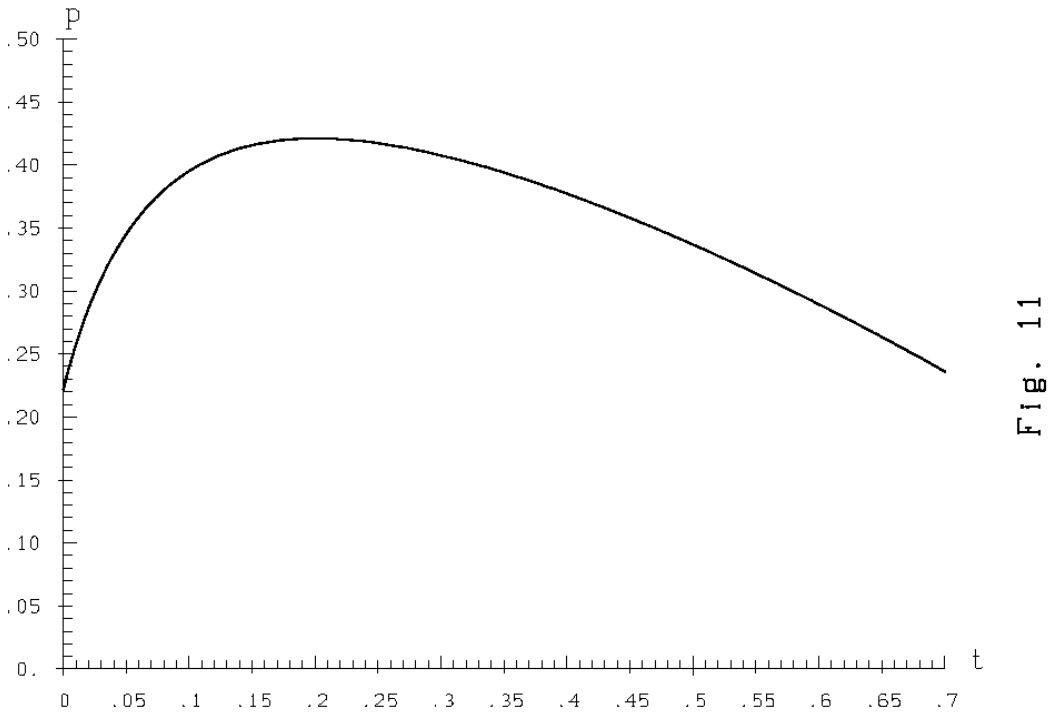

***Figure* 11.** A one humped price path with $0 < p < a_2$. Cases (b). The parameters chosen are $a_1 = 2.5$, $a_2 = 1$, $S = 10$, $M = 5$, $T = 0.7$.

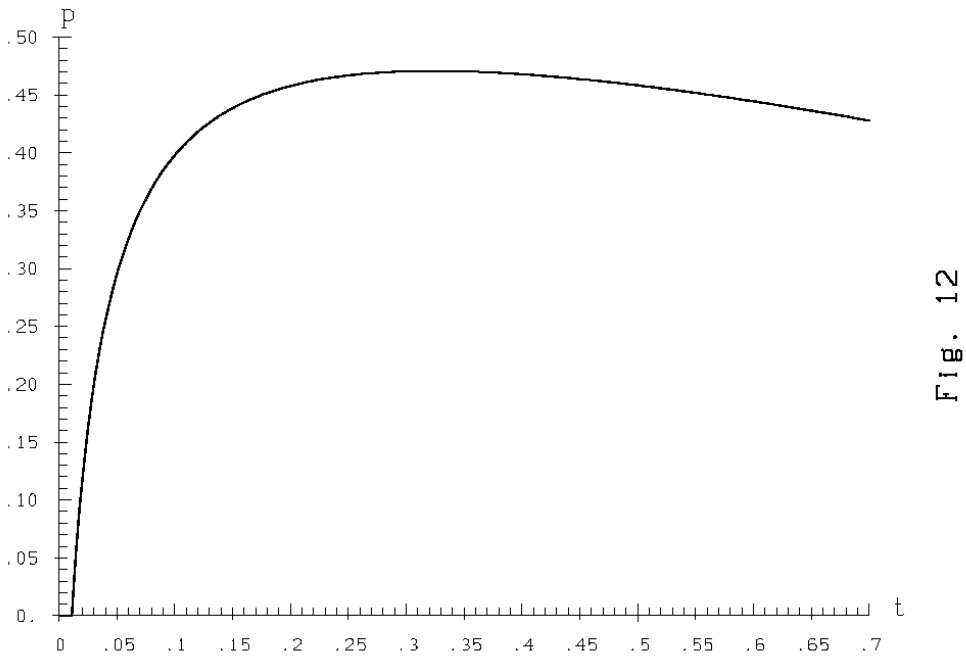

***Figure* 12.** A one humped price path starting with $p = 0$. Cases (a) - (b). The parameters chosen are $a_1 = 2.5$, $a_2 = 1$, $S = 10$, $M = 20$, $T = 0.7$.

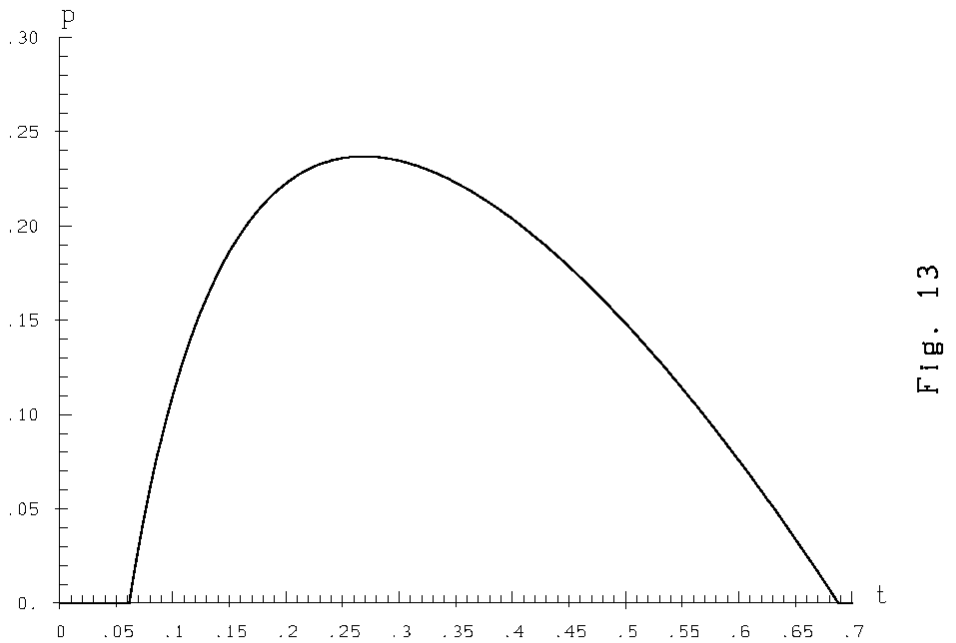

***Figure* 13.** A one humped price path starting and ending with $p = 0$. Cases (a) - (b) - (a). The parameters chosen are $a_1 = 2.5$, $a_2 = 1$, $S = 20$, $M = 5$, $T = 0.7$.

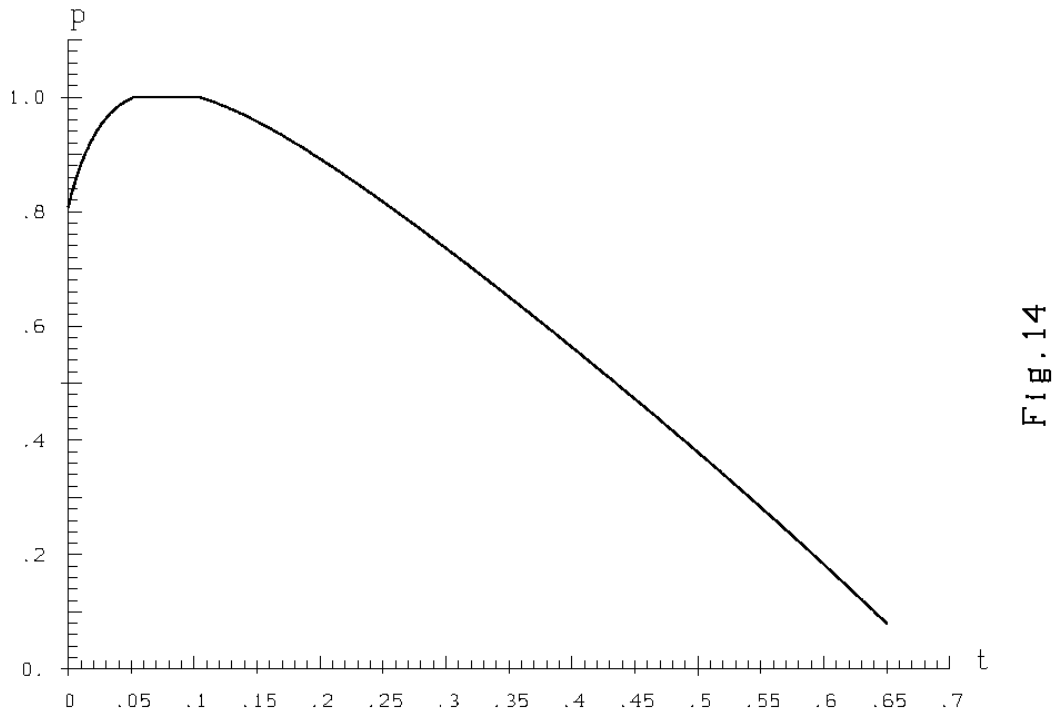

***Figure* 14.** A one humped price path spending some time at $p = a_2$. Cases (b) - (e) - (b). The parameters chosen are $a_1 = 5$, $a_2 = 1$, $S = 63$, $M = 5$, $T = 0.65$.

6. Concluding Remarks

We have formulated an optimal control problem to obtain the price over time in order to maximize profit from a product that diffuses in rich and poor populations. Depending on the relative sales to each of these populations, the product could be characterized as a status symbol or not. We find that depending on the parameters of the problem, price can be classified into one of three categories. In the first category, the product is a status symbol, whose initial price is prohibitive for the poor. Price declines over time to levels affordable to the poor. In the second category, the product is not a status symbol. The initial price is affordable to the poor and price declines over time. In the third category, the product again is not a status symbol, and price always remains affordable to the poor. However, here price increases initially, and eventually declines.

The optimal control problem solved here is non-linear with two state variables. The dynamics of the rich and the poor markets are asymmetric, which makes the problem interesting as well as complicated. A substantial part of the problem is addressed analytically. The results are obtained using both analytical and numerical means.

The model presented here considers the undiscounted case. An obvious extension is to consider a positive discount rate. This will substantially complicate the problem, and may result in price paths that may have more than one change in the sign of the derivative of price. However, we do not expect major changes to the results presented here.

A more interesting extension would consider future generations of products, with one generation being replaced by the next. In our model, we have recognized this issue through the salvage value S. A more realistic treatment would involve not only the optimal pricing of the successive products but also the timing of their introductions. Some work along these lines has been done in the context of machine maintenance and replacement (see Sethi and Thompson, Chapter 9).

7. Appendix:

***Proof of Proposition* 4:** Part (a) follows immediately from (19). Note that $\lambda_1 \geq a_1 + \frac{(Ma_2 - \lambda_2)F_1}{1-F_1}$ □ can also be expressed as $\lambda_1(1\text{-}F_1) + \lambda_2 F_1 \geq a_1(1\text{-}F_1) + Ma_2F_1$. Part (b) follows immediately from (20) and (21), since in this region there is no ambiguity. Part (c) is a result of the overlap of the conditions in (20) and (21). Solving this ambiguity by evaluating H for both values of p and – after some tedious calculations – identifying the global maximum yields (23) as the hairline case. Note that λ^* is clearly a value of λ_1 that satisfies $a_1 - 2a_2 \geq \lambda_1 \geq a_1 - 2a_2 - \frac{(Ma_2 + \lambda_2)F_1}{1-F_1}$ since $\sqrt{1+\frac{MF_1}{1-F_1}} < 1+\frac{MF_1}{1-F_1}$. Part (d) follows again from (20), (21), and Lemma 1, since in this region there is no ambiguity.

***Proof of Proposition* 5:** Part (a) follows immediately from (19). Part (b) follows immediately from (20). Part (d) follows again from (21).

Unlike in Region 2, now there is no overlap of the conditions in (20) and (21). Thus, Case (c) cannot occur. However, now there is a region when $a_1 - 2a_2 - \frac{(Ma_2 + \lambda_2)F_1}{1-F_1} > \lambda_1 > a_1 - 2a_2$, where neither the conditions in (20) nor in (21) are satisfied. In this case there are no interior local optima. Rather, the optimal p is at the kink a_2.

The following Lemma is needed for the proof of the next proposition:

Lemma 2:

$$-e^{1-x} \le \left[-\frac{5}{2}+2x-\frac{x^2}{2}\right] \qquad \forall x \le 1.$$

Proof: Follows from a Taylor series expansion of $-e^{1-x}$ up to order 2 around $x = 1$.

Proof of Proposition 6: For $p = 0$ the state equations reduce to

$$\frac{dF_1}{dt} = (1-F_1)(1-F_2)a_1, \tag{25}$$

$$\frac{dF_2}{dt} = F_1(1-F_2)a_2. \tag{26}$$

It is possible to give an upper bound for $F_1(t)$ in terms of $F_2(t)$ for solutions which stay in case (a). Along optimal solutions within case (a) the differential equation

$$\frac{dF_1}{dF_2} = \frac{a_1(1-F_1)}{a_2F_1}$$

has to hold, which implies $dF_2 = \frac{a_2F_1}{a_1(1-F_1)}dF_1$. Integrating yields

$F_2(t) = F_2(t_1) - \frac{a_2}{a_1}\left[F_1(t) - F_1(t_1) + \log\left(\frac{1-F_1(t)}{1-F_1(t_1)}\right)\right]$, which after some algebraic manipulations gives $F_1(t) = 1 - \frac{e^{1-F_1(t)}}{B(F_2(t))}$. From Lemma 2 it then follows that

$F_1(t) \le 1 + \frac{1}{B(F_2(t))}\left[-\frac{5}{2}+2F_1(t)-\frac{F_1(t)^2}{2}\right]$, which is equivalent to

$$0 \le B(F_2(t)) - \frac{5}{2} + F_1(t)(2-B(F_2(t))) - \frac{F_1(t)^2}{2}. \tag{27}$$

As $B(F_2(t)) \ge \frac{\exp(1-F_1(t_1))}{1-F_1(t_1)} \ge e$ holds, (27) is equivalent to (24).

Lemma 3: *Parts of solutions which follow Scenario 4 within the time interval (t_1,t_2) are given by*

$$F_1(t) = 1-(1-F_1(t_1))\left(\frac{k-t}{k-t_1}\right)^2, \tag{28}$$

$$F_2(t) \equiv F_2(t_1), \tag{29}$$

$$\lambda_1(t) = -a_1 - \frac{4}{(1-F_2(t_1))(t-k)}, \tag{30}$$

$$\lambda_2(t) = \lambda_2(t_1) + \frac{4(1-F_1(t_1))(t-t_1)}{(1-F_2(t_1))^2(k-t_1)^2}, \tag{31}$$

where $k = t_1 + \dfrac{4}{(\lambda_1(t_1)+a_1)(1-F_2(t_1))}$ *for all* $t \in (t_1, t_2)$. *Note that* $k > t_2$.

Proof: Under Scenario 4, i.e., for $a_2 < p < a_1$, the price is given by $p = \dfrac{a_1 - \lambda_1}{2}$. The state and the adjoint equations are

$$\frac{dF_1}{dt} = (1-F_1)(1-F_2)\left[\frac{a_1+\lambda_1}{2}\right], \tag{32}$$

$$\frac{dF_2}{dt} = 0, \tag{33}$$

$$\frac{d\lambda_1}{dt} = \left(\frac{a_1-\lambda_1}{2} + \lambda_1\right)(1-F_2)\left[a_1 - \frac{a_1-\lambda_1}{2}\right] = \left(\frac{a_1+\lambda_1}{2}\right)^2(1-F_2), \tag{34}$$

$$\frac{d\lambda_2}{dt} = \left(\frac{a_1-\lambda_1}{2} + \lambda_1\right)(1-F_1)\left[a_1 - \frac{a_1-\lambda_1}{2}\right] = \left(\frac{a_1+\lambda_1}{2}\right)^2(1-F_1). \tag{35}$$

Equation (29) is an immediate consequence of (33). The solution of (34) is then given by (30), where the constant k follows from the initial condition at $t = t_1$. From (32) we also obtain

$$F_1(t) = 1 - (1-F_1(t_1))e^{-\int_{t_1}^{t}(1-F_2(t_1))\left(\frac{a_1+\lambda_1(\tau)}{2}\right)d\tau} = 1 - (1-F_1(t_1))\left(\frac{k-t}{k-t_1}\right)^2,$$

where we have used

$$-\int_{t_1}^{t}(1-F_2(t_1))\left(\frac{a_1+\lambda_1(\tau)}{2}\right)d\tau = -\int_{t_1}^{t}\frac{2}{k-\tau}d\tau = 2\ln\left(\frac{k-t}{k-t_1}\right).$$

Furthermore, (35) reduces to

$$\frac{d\lambda_2}{dt} = \left(\frac{-2}{(1-F_2(t_1))(t-k)}\right)^2(1-F_1(t_1))\left(\frac{k-t}{k-t_1}\right)^2 = \frac{4(1-F_1(t_1))}{(1-F_2(t_1))^2(k-t_1)^2} \tag{36}$$

with its solution given by (31).

***Proof of Proposition* 7:**

(i) The time derivative of λ_1 along the border line $\lambda_1 = a_1 - 2a_2$ is given by

$$\dot{\lambda}_1\Big|_{\lambda_1=a_1-2a_2} = (1-F_2)(a_1-a_2)^2 > 0.$$

(ii) A transition from Scenario 2 to Scenario 4 can only occur in Region 1. In Region 1, Scenario 4 holds if, and only if,

$$\lambda_1(t) < \lambda^*(t) = a_1 - a_2 + \frac{\lambda_2(t)}{M} - \left(a_2 + \frac{\lambda_2(t)}{M}\right)\sqrt{1+\frac{MF_1(t)}{1-F_1(t)}},$$

which is equivalent to $g(t) < a_1 - 2a_2$ with

$$g(t) = \lambda_1(t) + \left(a_2 + \frac{\lambda_2(t)}{M}\right)\left(\sqrt{1+\frac{MF_1(t)}{1-F_1(t)}} - 1\right).$$

Note that *g(t)* is monotonically increasing under Scenario 4 in Region 1, as the solutions given by (28), (30) and (31) are monotonically increasing.

Let us assume that the transition from Scenario 2 to Scenario 4 occurs at time t_1. This, however, is not possible as it would imply $g(t_1) = a_1 - 2a_2, g(t) < a_1 - 2a_2$ for all $t \in (t_1, t_1 + \varepsilon)$ for some $\varepsilon > 0$, which is a contradiction to the monotonicity of *g*.

Proof of Proposition 8: Scenario 4 holds at time *T* if $S < S^* = a_1 - a_2\left[1+\sqrt{1+\frac{MF_1}{1-F_1}}\right]$.

Obviously, $S^* < a_1 - 2a_2$, and therefore $0 \le S$ together with $S < S^*$ require $a_1 \ge 2a_2$.

Proof of Proposition 9: Apply the result of Lemma 3 to the case $t_1 = 0, t_2 = T$ together with the boundary conditions $F_1(0) = F_2(0) = 0, \lambda_1(T) = S, \lambda_2(T) = 0.$ Equation (30) together with $F_2(0)=0$ then yields $\lambda_1(t) = -a_1 - \frac{4}{t-k}$, where $k = T + \frac{4}{S+a_1}$ follows from $\lambda_1(T) = S$, and

$p(t) = \frac{a_1 - \lambda_1(t)}{2} = a_1 + \frac{2}{t-k}$. Inserting *k* into (28) gives $F_1(t) = 1 - \left(\frac{(S+a_1)(T-t)+4}{(S+a_1)T+4}\right)^2$.

The solution for $\lambda_2(t)$ follows from the differential equation (36) together with the transversality condition $\lambda_2(T)=0$

Proof of Proposition 10: Proposition 10 is an immediate consequence of Proposition 7. Solutions which end in Scenario 4 has to completly lie in Scenario 4 as a transition from Scenario 3 as well as from Scenario 2 to Scenario 4 are not possible.

Proof of Proposition 11: Since from the transversality condition we have $\lambda_2(T) = 0$, it is sufficient to show that $\lambda_2(t) = 0$ implies $\dot{\lambda}_2(t) > 0$. Assume $\lambda_2(t) = 0$ at some *t*. Corresponding to the different scenarios, we consider the following cases:

- Scenario 1

  $\dot{\lambda}_2\big|_{\lambda_2=0} = (1-F_1)\lambda_1 a_1 > 0$ as in scenario 1 and $\lambda_1 \geq a_1 + Ma_2 \dfrac{F_1}{1-F_1} > 0.$

- Scenario 2:

  At $\lambda_2 = 0$, this scenario occurs for

  $S^* = a_1 - a_2\left(1+\sqrt{1+\dfrac{MF_1}{1-F_1}}\right) < \lambda_1 < a_1 + a_2\dfrac{MF_1}{1-F_1}$. The price is given by

  $p\big|_{\lambda_2=0} = \dfrac{(1-F_1)(a_1-\lambda_1)+F_1Ma_2}{2(1-F_1)+2MF_1}$. Defining $\alpha = \dfrac{F_1M}{1-F_1}$, we get

  $\dot{\lambda}_2\big|_{\lambda_2=0} = (1-F_1)[(p+\lambda_1)(a_1-p)+\alpha p(a_2-p)]$ with $p = \dfrac{a_1-\lambda_1+\alpha a_2}{2(1+\alpha)}$.

  Inserting for *p* yields $\dot{\lambda}_2\big|_{\lambda_2=0} = \dfrac{1-F_1}{4(1+\alpha)} f(\lambda_1)$ with

  $f(\lambda_1) = \lambda_1^2 + 2\lambda_1(a_1+2\alpha a_1-\alpha a_2)+(a_1+\alpha a_2)^2$. The quadratic equation $f(\lambda_1)=0$ has real roots at $-(a_1+2\alpha a_1-\alpha a_2)\pm 2\sqrt{\alpha(1+\alpha)a_1(a_1-a_2)}$. Since $f(\lambda_1)$ is convex, it is sufficient to show that

  $$S^* \geq -(a_1+2\alpha a_1-\alpha a_2)+2\sqrt{\alpha(1+\alpha)a_1(a_1-a_2)}\,, \tag{37}$$

  as this implies $f(\lambda_1) > 0 \quad \forall \lambda_1 > S^*$. (37) is equivalent to

  $$(2a_1-a_2)(1+\alpha)-a_2\sqrt{1+\alpha}-2\sqrt{\alpha(1+\alpha)a_1(a_1-a_2)} \geq 0. \tag{38}$$

  Inequality (38) holds, since the LHS of (38) is equal to

  $$\sqrt{(1+\alpha)}\left(1+\sqrt{1+\alpha}\right)\left(\sqrt{(a_1-a_2)}-\frac{\sqrt{\alpha a_1}}{1+\sqrt{1+\alpha}}\right)^2.$$

- Scenario 4:

$$\dot{\lambda}_2\Big|_{\lambda_2=0} = (1-F_1)\left(\frac{a_1+\lambda_1}{2}\right)^2 > 0\,.$$

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