OPTIMAL INVESTMENT FACING POSSIBLE ACCIDENTS

Richard F HARTL
Institute of Management, University of Vienna,
Brünnerstraße 72, A-1210 Vienna, Austria

Peter M. KORT
Department of Econometrics and CentER, Tilburg University (KUB),
P.O. Box 90153, NL-5000 LE Tilburg, Netherlands

Andreas J. NOVAK
Institute of Statistics and Computer Science, University of Vienna,
Universitätsstraße 5, A-1010 Vienna, Austria

Abstract:

This paper studies the optimal behavior of a firm over time that faces the probability of causing an environmental disaster by its activities. Here we can think of explosions in the chemical industry, oil tankers that lose oil, etc. In the static literature a distinction is made between small firms and large firms. Small firms are called undercapitalized in the sense that the share holders are not willing to pay for the accident because of the reason that the accidental damage exceeds the value of the firm. As soon as this happens the firm is declared bankrupt. This gives an incentive to spend part of the safety budget on distributing dividends to the shareholders under the motto: pay dividends as long as it is still possible.

In this paper we extend the static framework to a dynamic one. The major reason is to find out under what circumstances it can be optimal for a small firm to expand in order to become a large firm that has the means to fully pay for the damage caused by an environmental accident. Admittedly, this paper is only a first step in reaching this goal. Here, we determine the optimal safety expenditures and the convergence behavior of long run investment policies.

1 Introduction

Veld (1996) analyses the behavior of a firm that faces the risk of causing an environmental disaster by its activities (see also Beard (1990)). Think of e.g. the oil industry where tankers can lose oil.

If the costs of a disaster exceed the value of the shares then shareholders are not willing to pay for it so that bankruptcy results. The firm has the possibility to carrying out safety expenditures which reduce the probability of the occurrence of an accident. The cost of the disaster is a random variable whose distribution cannot be influenced by the firm.

Veld's model is static and distinguishes between small and large firms. For small firms an accident causes bankruptcy with positive probability and it turns out that it is optimal for such a firm to invest less in safety than is socially optimal. Instead, it pays more dividends to the shareholders. On the other hand large firms survive all accidents because the value of the shares always exceeds the costs of a disaster. Therefore it pays to invest in safety at the socially optimal level.

We know from several dynamic models of the firm (Van Hilten et al., 1993) that a firm pays dividend only when it is sufficiently large while small firms retain all their earnings to use it for investment. This result contrasts the findings of Veld. This might be caused by the fact that in Veld's static model the size of the firm is exogenously given and the firm cannot grow or contract.

The purpose of this research is to incorporate the possibility of firm growth into Veld's framework and to find out under what scenario a small firm prefers to grow first and pay dividends later rather than paying dividends right away, as was found in Veld (1996). This paper must be seen as a first step taken in order to reach this goal.

The contents of the paper is as follows. Section 2 lists the model variables and presents a function that specifies the probability of the firm going bankrupt as a function of safety expenditures and productive capital stock. In section 3 the dynamic model of the firm is presented in which the optimal safety expenditures and investment level must be determined over time in order to maximize the discounted dividend stream. Section 4 contains two subsections. In the first subsection the optimal level of safety expenditures is determined, whereas in the second subsection we give economic interpretations and study the stability of the steady state(s).

2 Definitions

We consider a firm whose operations might result in accidents, think of e.g. the oil industry where tankers can lose oil which results in environmental disasters. Here we assume that per unit of capital stock (i.e. per tanker) the probability that an accident occurs is μ.

As in Veld (1996) we assume the capital market to be perfect. An argument along the lines of Brander and Spencer (1989) shows that firms strictly prefer equity over debt if creditors cannot verify expenditures on accident prevention. Therefore, the firm under consideration is all-equity financed and managed such that the value of the firm for the shareholders is maximized. If a firm has a capital stock K, it has K shares outstanding. Each share has a market value of v. An important extension to the model of Veld (1996) is that, where in Veld (1996) the firm immediately distributes all profits as dividends, in our model the firm can also retain its earnings and use it for investments to increase its capital stock. Like in Veld (1996) the shareholders are risk-neutral and face a constant opportunity cost of investment, equal to the return r on a riskless bond.

Each unit of capital gives rise to accidents at times generated by a Poisson process with mean μ. These accidents do not damage the firm's capital, i.e., they are of the nature of chemical spills rather than explosions. Accidents happen independently across units of capital. As a result, the times at which the firm, having a capital stock K, experiences accidents can be treated as if they are generated by a Poisson process with mean $K\mu$.

Financially, every disaster costs $A(S)$ and if this exceeds the value vK of the shares then shareholders are not willing to pay for it so that bankruptcy results. Firms are assumed to have no insurance for these disasters. Theoretically spoken, this assumption can be defended by arguing that the firm has no incentive to buy insurance, because the shareholders are risk neutral (Veld (1996)).

The firm has the possibility to reduce $A(S)$ by carrying out safety expenditures so that the function A is decreasing in S.

We will use the following notation:

K ... productive capital stock (state)

$R(K)$ ... revenue; $R' > 0$, $R'' < 0$, $R(0) = 0$

I ... investment in the productive capital stock (control)

$c(I)$ ... costs of investment; $c' > 0$, $c'' > 0$, $c(0) = 0$

S ... spending on safety (control)

$A(S)$ ... costs of an accident given safety S, $A' < 0$, $A'' > 0$, $A(\infty) = 0$, $A(0)$ "large"

w ... unit cost of S (parameter)

μ ... probability that an accident occurs per unit of time *per unit of capital stock*

$F(t)$ ... probability that bankruptcy has occurred before time t (CDF) (state)

$x = A(S) - vK$ ... amount by which the damage $A(S)$ of an accident exceeds the value of the shares

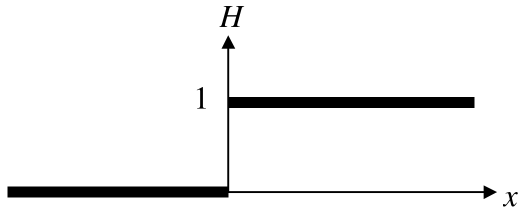
v ... value of one share

If at the moment of an accident $x > 0$, i.e., $A(S) > vK$ at any instance, then the firm goes bankrupt, while for $x < 0$ the value of the firm exceeds the damage of the accident. Then, the firm simply collects the damage payments $A(S)$ directly from the shareholders in the form of cash contributions. Neither the level of K nor the market valve v of its outstanding shares is then affected by such an accident (Veld(1996)).

Thus, the probability (rate) that the firm goes bankrupt due to the occurrence of an accident with damage $A(S)$ (given that it has survived so far) is $H(A(S) - vK)$, where H is the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases} \tag{1}$$

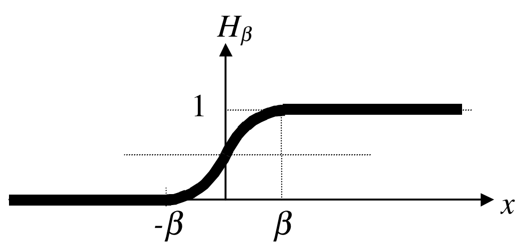
Figure* 1*: Heaviside function H.



In what follows, we solve the model by applying Pontryagin's maximum principle. This requires the use of continuous functions so that we have to approximate H by a continuous function H_β which approximates H arbitrarily well as $\beta \to 0$, i.e., $\lim_{\beta \to 0} H_\beta(x) = H(x)$:

$$H_\beta(x) = \begin{cases} 0 & \text{for } x \leq -\beta \\ \frac{1}{2}\left[1 + \frac{x}{\beta}\right]^2 & \text{for } -\beta < x < 0 \\ 1 - \frac{1}{2}\left[1 - \frac{x}{\beta}\right]^2 & \text{for } 0 \leq x < \beta \\ 1 & \text{for } x \geq \beta \end{cases} \tag{2}$$

Figure* 2*: A smooth function H_β.



The economic interpretation for $0 < H < 1$ in the interval $(-\beta, \beta)$ is that there is some uncertainty whether the value of the firm is correctly measured in vK. Thus banks might be willing to help the firm to pay for the damage in case this damage is not much larger than the value of the shares. On the other hand a damage just below the value of the shares can also cause bankruptcy because some of the shareholders might not be willing to pay for the damage.

3 Model formulation

It is assumed that the firm behaves as if it maximizes the discounted value of the expected future profit stream*:

$$\max_{S,I} \int_0^\infty e^{-rt} \left[R(K) - wS - C(I) - \mu K A(S)\right]\left[1 - F\right] dt \tag{3}$$

The cash inflow equals the revenue the firm obtains by selling its products on the market, which is given by $R(K)$. The cash outflow consists of three elements. First, we have the money spent on safety, wS, as a second element we have the investment costs, $C(I)$, and the third element consists of the expected damage of an accident for one capital good, $\mu A(S)$, multiplied by the number of capital goods, K. Of course, profits only count as long as the firm is not bankrupt, the probability of which is given by $1\text{-}F$.

Productive capital stock increases with investments and decreases through depreciation, where the latter takes place in an exponential way:

$$\dot{K} = I - aK, \quad K(0) = K_0 \tag{4}$$

Since F(t) denotes the probability that bankruptcy has occurred before time t, the probability that bankruptcy occurs precisely at time t is given by $\dot{F}$. The latter probability is equal to the probability that bankruptcy did not occur before time t, $1\text{-}F$, times the probability that an accident exactly occurs at time t for a particular unit of capital stock, μ, times the total of capital goods, K, times the probability that the firm goes bankrupt at the moment of an accident with damage $A(S)$ which is $H(A(S)\text{-}vK)$. All this leads to the following state equation for F:

$$\dot{F} = \left[1 - F\right]\mu K H(A(S) - vK), \quad F(0) = 0 \tag{5}$$

The problem is to determine at each point of time the value of the control variables $S(t)$ and $I(t)$ such that objective (3) is maximized, subject to the state equations (4) and (5).

In what follows, we will use the smooth version H_β of the Heaviside function from (2) but for notational convenience, we will simply call it H.

4 The Solution:

In order to obtain optimal firm behavior we start out by determining necessary conditions that need to be fulfilled in case of an optimal solution. To obtain these conditions we apply Pontryagin´s maximum-principle. To do so we first write down the Hamiltonian:

$$\mathbf{H} = \left[R(K) - wS - C(I) - \mu K A(S)\right]\left[1 - F\right] + \lambda_1\left[I - aK\right] + \lambda_2\left[1 - F\right]\mu K H(A(S) - vK), \tag{6}$$

Then the maximum principle conditions are

$$\mathbf{H}_I = -C'(I)\left[1 - F\right] + \lambda_1 = 0 \tag{7}$$

* Parentheses $f(x)$ always indicate function f evaluated at x while brackets $f[x]$ indicates a multiplication of f with expression x.

$$\mathbf{H}_S = [-w - \mu KA'(S)][1-F] + \lambda_2[1-F]\mu KH'(A(S)-vK)A'(S) = 0 \quad \text{i.e.}$$
$$-w - \mu KA'(S) + \lambda_2 \mu KH'(A(S)-vK)A'(S) = 0 \quad \text{or} \quad F = 1 \tag{8}$$

$$\dot{\lambda}_1 = r\lambda_1 - \mathbf{H}_K = [r+a]\lambda_1 - [R'(K) - \mu A(S)][1-F]$$
$$-\lambda_2[1-F]\mu H(A(S)-vK) + v\lambda_2[1-F]\mu KH'(A(S)-vK) \tag{9}$$

$$\dot{\lambda}_2 = r\lambda_2 - \mathbf{H}_F = r\lambda_2 + R(K) - wS - C(I) - \mu KA(S) + \lambda_2 \mu KH(A(S)-vK)\,. \tag{10}$$

While it is difficult to verify, that the following limiting transversality condition is also part of the necessary conditions, it is customary in economic control problems to look mainly for solutions satisfying it, e.g. bounded solutions and in particular those converging to an equilibrium.:

$$\lim_{t\to\infty} e^{-rt}\lambda_1(t) = \lim_{t\to\infty} e^{-rt}\lambda_2(t) = 0 \tag{11}$$

4.1 Optimal safety expenditures

From expression (2) we obtain that the bankruptcy risk after occurrence of an accident is zero if it holds that

$$x \le -\beta \quad \textit{i.e.,} \quad A(S) - vK \le -\beta\,. \tag{12}$$

From (12) we derive that for $K > \beta/v$ the firm is always able to prevent bankruptcy by spending enough money on safety. At a given point of time bankruptcy is prevented for sure if it holds that

$$S \ge A^{-1}(vK - \beta). \tag{13}$$

On the other hand we can obtain from expression (2) that an accident leads to bankruptcy with probability 1 if it holds that

$$x \ge \beta \quad \textit{i.e.,} \quad A(S) - vK \ge \beta\,, \tag{14}$$

which can be rewritten into

$$S \le A^{-1}(vK + \beta). \tag{15}$$

Expressions (13) and (15) enable us to specify regions in the (K, S)-plane where

(Region I) bankruptcy does not take place after an accident,

(Region II) bankruptcy takes place with positive probability after an accident, and

(Region III) bankruptcy takes place for sure after an accident.

These regions are depicted in Figure 3.

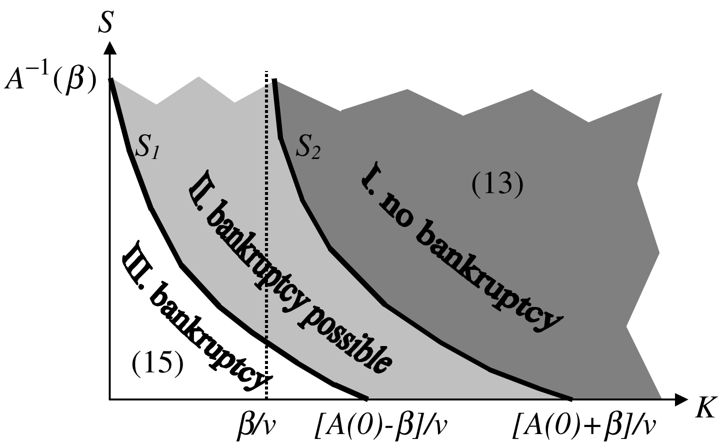


***Figure* 3.** Possibilities after occurrence of accident depending on S and K.

From this figure we conclude that, in order to prevent bankruptcy, the firm needs to spend less money on safety when productive capital stock is large. This is because the value of the firm increases proportionally with K, and as long as this value exceeds the damage $A(S)$ by an amount more than β, the shareholders prefer to pay for the damage rather than letting the firm go bankrupt.

In Regions I and III we have $H'(A(S) - vK) = 0$, while in Region II we have $H'(A(S) - vK) > 0$; cf. Figure 2. In what follows, we will use the notation

$$S_1 = A^{-1}(vK + \beta),\ S_2 = A^{-1}(vK - \beta),$$

In the regions with $H'(A(S) - vK) = 0$, equation (8) for F < 1 is easy to solve:

$$- w - \mu KA'(S) = 0, \quad \text{i.e.,} \quad S = \tilde{S} \text{ with}$$

$$\tilde{S} = (A')^{-1}(-w/\mu K) \tag{8'}$$

In order to obtain the global maximum of **H**, it is convenient to consider three different cases.

Proposition* 1*: Let $\tilde{S}$ be the solution of (8'), i.e. the value, where the Hamiltonian **H** assumes its maximum, if the last term $\lambda_2[1 - F]\mu KH(A(S) - vK)$ is absent.

- **Case a:** If $\tilde{S} > S_2$, then $\tilde{S}$ is the global maximum of the Hamiltonian **H** (in **Region I)**
- **Case b:** If $S_1 < \tilde{S} < S_2$ then the Hamiltonian **H** assumes its global maximum in the (open) interval ($\tilde{S}$, S_2) in **Region II**.
- **Case c:** If $\tilde{S} < S_1$ then $\tilde{S}$ is one local maximum of the Hamiltonian **H** in **Region III** while there can be another in the (open) interval (S_1, S_2), i.e., in **Region II**. If there are two local maxima, then we have to evaluate **H** at each to find out which one is the global one.

The proof is obvious, since the first part of **H**, $[R(K) - wS - C(I) - \mu KA(S)][1 - F] + \lambda_1[I - aK]$, is strictly concave in S if $F < 1$, while the second part $\lambda_2[1 - F]\mu KH(A(S) - vK)$ is shaped like H_β as sketched in Fig. 2.

Proposition 1 tells us that the optimal safety level is given by (8´) in the non bankruptcy region (I) and possibly in the bankruptcy region (III). In this case it holds that marginal costs of S (which equal w) equals marginal benefits, where the latter equals the reduction in expected damage due to an additional unit of S. Furthermore, from (8´) it is easy to obtain that the firm spends more money on safety if capital stock is large which makes economically sense.

For reasons of illustration let us study optimal safety behavior under the following functional form:

$$A(S) = \alpha e^{-\gamma S}, \qquad A'(S) = -\gamma\alpha e^{-\gamma S} = -\gamma A(S) \tag{16}$$

Then, after substitution of (16) into (8), (13), and (15) it is possible to derive for what values of K which region occurs:

Case a. $(A')^{-1}(-w/\mu K) > A^{-1}(vK - \beta)$, i.e., $\quad K > \dfrac{\beta + \sqrt{\beta^2 + 4\frac{vw}{\mu\gamma}}}{2v} = K_1$

Case b. $K_2 < K < K_1$

Case c. $(A')^{-1}(-w/\mu K) < A^{-1}(vK + \beta)$, i.e., $\quad K < \dfrac{-\beta + \sqrt{\beta^2 + 4\frac{vw}{\mu\gamma}}}{2v} = K_2$

Hence, if the capital stock exceeds K_1, then it is optimal for the firm to spend so much on safety, that bankruptcy risk is zero after possible occurrence of an accident; the optimal S is given by (8´).

In Region II the optimal safety level has to satisfy

$$-w-\mu KA'(S)+\lambda_2\mu KH'(A(S)-vK)A'(S)=0$$

which (because of H' = (1 - |x|/β)/β) can be rewritten into

$$-w + \mu K\gamma\alpha e^{-\gamma S} - \lambda_2\mu K[1 - |[A(S)-vK]/\beta|]\gamma\alpha e^{-\gamma S}/\beta = 0.$$

This means: $A = \alpha e^{-\gamma S}$ is the solution of

$$-w + \mu K\gamma A - \lambda_2\mu K[1 - |[A - vK]/\beta|]\gamma A/\beta = 0 \tag{17}$$

By investigating the two subcases $A(S) - vK < 0$ and $A(S) - vK > 0$ one obtains the optimal value of S, by : $A = \alpha e^{-\gamma S}$ and[1]

$$A=\frac{K\left[\lambda_2[\beta-vK]-\beta^2\right]+\sqrt{K^2\left[\lambda_2[\beta-vK]-\beta^2\right]^2-\frac{4w\beta^2K\lambda_2}{\mu\gamma}}}{-2K\lambda_2} \tag{18}$$

if $A < vK$. This inequality becomes

$$\left[\lambda_2[\beta-vK]-\beta^2\right]+\sqrt{\left[\lambda_2[\beta-vK]-\beta^2\right]^2-\frac{4w\beta^2\lambda_2}{\mu\gamma K}}<-2vK\lambda_2,$$

which leads to

$$\left[\lambda_2\beta-\beta^2\right]+\sqrt{\left[\lambda_2[\beta-vK]-\beta^2\right]^2-\frac{4w\beta^2\lambda_2}{\mu\gamma K}}<-vK\lambda_2.$$

From this inequality we conclude that the damage of a possible accident is given by (18) in case it holds that $K_4 < K \le K_1$ with

$$K_4=\sqrt{\frac{\beta w}{v\mu\gamma[\beta-\lambda_2]}}.$$

Unfortunately, as we will see later, for values of K close to K_4 the value from (18) need not be the global maximum, but can be a local maximum.

In the other subcase, $A(S) - vK > 0$, one can compute the optimal value of S, by: $A = \alpha e^{-\gamma S}$ and

$$A=\frac{-K\left[\lambda_2[\beta+vK]-\beta^2\right]\pm\sqrt{K^2\left[\lambda_2[\beta+vK]-\beta^2\right]^2+\frac{4w\beta^2K\lambda_2}{\mu\gamma}}}{-2K\lambda_2}$$

It can be shown that the larger root is a local minimum so that we arrive at

$$A=\frac{-K\left[\lambda_2[\beta+vK]-\beta^2\right]-\sqrt{K^2\left[\lambda_2[\beta+vK]-\beta^2\right]^2+\frac{4w\beta^2K\lambda_2}{\mu\gamma}}}{-2K\lambda_2} \tag{19}$$

[1] Note that the expression only makes sense if λ_2 is negative, because otherwise A would become negative. This makes sense because λ_2 is the shadow price of bankruptcy. Note also, that the second root of (17) is negative, which leads to expression (18).

In order for the expression under the square root to be positive, K must satisfy $K > K_3$ with K_3 being the solution of

$$K^3[\lambda_2 v]^2 - 2K^2\lambda_2 v\beta[\beta-\lambda_2] + K\beta^2[\beta-\lambda_2]^2 + \frac{4w\beta^2\lambda_2}{\mu\gamma} = 0$$

It can easily be shown that the solution of this equation is unique and positive.

It remains to verify the conditions $0 < A - vK < \beta$. After some algebra these conditions reduce to

$K_3 < K < K_4$ for $\lambda_2 < -\beta^2\mu\gamma K_2/w$, and

$K_2 < K < K_4$ for $-\beta^2\mu\gamma K_2/w < \lambda_2 < 0$.

Figures 4 and 5 illustrate the occurrence of the maxima in the various regions[2]. In Figure 4, which represents the case $\lambda_2 < -\beta^2\mu\gamma K_2/w$, it is seen that between K_3 and K_2 there are two local maxima of the Hamiltonian, one in the Region III and the other in the Region II. For the parameters chosen, it was checked numerically that there is a threshold level of K between K_3 and K_4 so that the solution in Region II (III) is the optimal one for K larger (smaller) than this threshold value. It could also happen that this threshold occurs between K_4 and K_2; see the remark following equation (18).

***Figure* 4.** The Hamiltonian maximizing value of S for $\lambda_2 < -\beta^2\mu\gamma K_2/w$.

[2] In Figure 4 we have chosen the parameters $\alpha = 5$, $\gamma = 1$, $w = 0.05$, $\mu = 0.1$, $\beta = 0.5$, $v = 0{,}5$ and $\lambda_2 = -1.2$ for performing the numerical calculations. In Figure 5 we have selected the same parameters except for $\lambda_2 = -0.25$.

***Figure* 5.** The Hamiltonian maximizing value of S for $-\beta^2\mu\gamma K_2/w < \lambda_2 < 0$.

Figure 5 represents the case $-\beta^2\mu\gamma K_2/w < \lambda_2 < 0$ where there are no local maxima and the optimal S is a continuous function of K for fixed value of λ_2.

***Figure* 6.** The values of K_1, to K_4 as functions of λ_2.

Figure 6 shows the dependence of K_1, to K_4 on the value of λ_2 (same parameters as in Figure 4 and 5). Note that K_1 and K_2 do not depend on λ_2 while K_3 and K_4 are increasing functions of λ_2 (K_3 is increasing only in the relevant region $\lambda_2 < -\beta^2\mu\gamma K_2/w$).

4.2 Optimal long run investment behavior

As a first attempt to study the optimal long run investment behavior we determine the steady state(s) in the no bankruptcy region (see Region I in Figure 3). After having done so we give economic interpretations and study the stability of the steady state(s). In order to be able to actually arrive in this region, there has to exist a certain positive probability that the firm did not go bankrupt before the arrival in region (I), which implies that $F < 1$. From (8) we obtain that $F < 1$ in turn implies that $K > 0$. Being in the no bankruptcy region by definition means that the probability of going bankrupt now is zero, which leads to $\dot{F} = 0$. Due to $F < 1$, $K > 0$ and $\dot{F} = 0$, we obtain from (5) that $H(A(S)-vK) = 0$. This implies via (2) that $vK > A(S)+\beta$, and this means that the value of the firm is that large that the shareholders are always willing to pay for the accident.

In the steady state it holds that $\dot{K} = \dot{F} = 0$ and $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$. Combining this with the above conclusions we obtain from expressions (4) and (7) to (10) that in a possible steady state we have

$$I = aK, \tag{20}$$

$$\lambda_1 = C'(I)[1-F] \tag{21}$$

$$-w - \mu KA'(S) = 0 \tag{22}$$

$$[r+a]\lambda_1 = [R'(K) - \mu A(S)][1-F] \tag{23}$$

$$r\lambda_2 + R(K) - wS - C(I) - \mu KA(S) = 0 \tag{24}$$

From (21) and (23) we can eliminate λ_1 and obtain

$$[r+a]C'(I)[1-F] = [R'(K) - \mu A(S)][1-F], \text{ i.e.,}$$

$$[r+a]C'(aK) + \mu A(S) = R'(K) \tag{25}$$

which together with (22) gives a system of two equations for the two variables $\hat{K}$ and $\hat{S}$.

Since it is - in general - not clear how many solutions this system of equations has, we will consider a special case in the next subsection in which we will specify the model functions further. But before let us make a few observations:

Remarks:

a) Equation (25) implies that marginal revenue of K, $R'(K)$, equals marginal costs, which consists of marginal investment costs, $[r+a]C'(aK)$, and the increase in expected damage due to an additional unit of K.

b) For a given solution $\hat{K}$ and $\hat{S}$ of (22) and (25), $\hat{I}$ can be obtained from (20) and λ_2 can be computed from (24). This equilibrium, however, is an equilibrium for all $\hat{F} < 1$ and, depending on the value of $\hat{F}$, λ_1 can be computed from (21) or (23). This means, that for given values of $\hat{K}$ and $\hat{S}$ one has a *one-dimensional family of equilibria*, since $\hat{F}$ (and therefore also λ_1) is *not unique*.

The reason is that in this equilibrium F plays no role at all. K is that large that, given that the firm still exists (which is not sure if $F > 0$), the probability of bankruptcy coming up now is zero, $\dot{F} = 0$. Hence, if the equilibrium happens to be a saddle point the firm will never go bankrupt once the trajectory has passed into this equilibrium. This also holds after a marginal change of K.

c) A necessary condition that the above values really constitute an equilibrium in the no bankruptcy region is that $H(A(S) - vK) = 0$, as observed in the beginning of this section (see subsection 4.1 for more details). This means that $A(S) - vK < -\beta$, which can also be formulated as $A(\hat{S}) + \frac{vw}{\mu A'(\hat{S})} < -\beta$ (cf . (22)). In combination with $A'(S) < 0$ this yields:

$$\left[A(\hat{S}) + \beta\right]A'(\hat{S})\mu + vw > 0 \tag{26}$$

which represents a lower bound on S, since the l.h.s. of (26) is strictly increasing in S.

4.2.1 Example

For the rest of this section we specify the model functions as follows:

$$C(I) = cI^2/2, \quad C'(I) = cI, \quad C''(I) = c; \quad \text{(with } c > 0) \tag{27}$$

$$A(S) = \alpha e^{-\gamma S}, \quad A'(S) = -\gamma\alpha e^{-\gamma S}, \quad A''(S) = \gamma^2\alpha e^{-\gamma S}; \quad \text{(with } \alpha > 0, \gamma > 0) \tag{28}$$

$$R(K) = K^b, \quad R'(K) = bK^{b-1}, \quad R''(K) = b[b-1]K^{b-2}; \quad \text{(with } 0 < b \leq 1) \tag{29}$$

Now (22) and (25) become

$$-w + \mu K\gamma\alpha e^{-\gamma S} = 0 \tag{22'}$$

$$\left[r + a\right]caK - bK^{b-1} + \mu\alpha e^{-\gamma S} = 0 \tag{25'}$$

We can eliminate S to obtain one equation for K:

$$\left[r + a\right]caK - bK^{b-1} + \frac{w}{\gamma K} = 0 \tag{30}$$

Proposition 2: Equation (30) has either 2 roots or no solution at all (except for hairline cases). More precisely,

$$\text{if } F(\tilde{K}) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \text{ then there are/is } \begin{cases} 2 \text{ equilibria} \\ 1 \text{ equilibrium} \\ \text{no equilibrium} \end{cases},$$

where

$$F(K) = \left[r + a\right]ca\gamma K^2 - \gamma bK^b + w \tag{31}$$

and

$$\tilde{K} = \left[\frac{b^2}{2\left[r + a\right]ca}\right]^{\frac{1}{2-b}} \tag{32}$$

is the local (and global) minimum of $F(K)$. The equilibrium value(s) of K is (are) obtained by setting $F(K) = 0$.

Proof: Follows from the fact that $F(K)$ is strictly convex with $F(0) = w > 0$. See Figures 7 to 9 below. Note that $F(K)$ is simply the l.h.s. of (30) multiplied by γK.

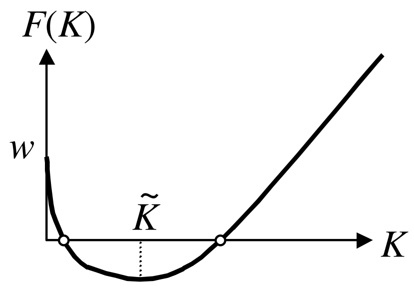


***Figure* 7:** Two equilibria.

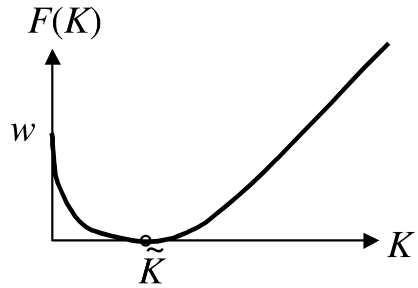


***Figure* 8:** The hairline case of one equilibrium.

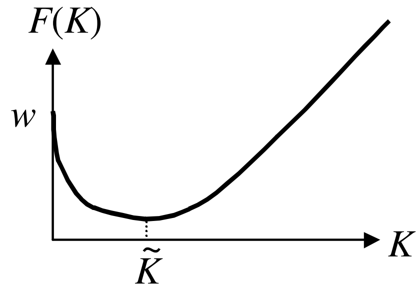


***Figure* 9:** No equilibrium.

4.3 Stability of the equilibria

Here it is important to note that the above results are only valid for Region I (no bankruptcy), which implies that $\hat{S}$ must be sufficiently large so that expression (26) is satisfied. In order to find out whether a particular equilibrium is stable or not, we first transform the 4-dimensional canonical system into a 3-dimensional system in *K, I*, and λ_2 by eliminating λ_1 and deriving a DE for *I*:

$$\dot{K} = I - aK$$

$$\dot{I} = \left[r + a + \mu K H(A(S) - vK)\right] I - \tfrac{1}{c}\left[R'(K) - \mu A(S)\right] - \tfrac{\lambda_2}{c} \mu H(A(S) - vK) + v \tfrac{\lambda_2}{c} \mu K H'(A(S) - vK)$$

$$\dot{\lambda}_2 = r\lambda_2 + R(K) - wS - C(I) - \mu K A(S) + \lambda_2 \mu K H(A(S) - vK)\,.$$

For $H = H' = 0$ this can be further simplified to

$$\dot{K} = I - aK \tag{33}$$

$$\dot{I} = \left[r + a\right] I - \tfrac{1}{c}\left[R'(K) - \mu A(S)\right] \tag{34}$$

$$\dot{\lambda}_2 = r\lambda_2 + R(K) - wS - C(I) - \mu K A(S)\,. \tag{35}$$

As already mentioned, the stability of an equilibrium in this system with given values of *K*, *S*, *I*, λ_1, and λ_2 does not depend on the value of *F*.

The Jacobian matrix of system (33) – (35) is:

$$\begin{pmatrix} -a & 1 & 0 \\ -\dfrac{R''}{c} - \dfrac{w}{c\gamma K^2} & r + a & 0 \\ R' - \dfrac{w}{\gamma K} & -cI & r \end{pmatrix}, \tag{36}$$

where it is taken into account that *S* is a function of *K* as given in (22). Clearly, *r* is one of the eigenvalues. Furthermore, we can obtain the following important result for the most interesting case with two equilibria:

Proposition 3: Let us consider the case with two equilibria, and let us denote the corresponding capital stocks by $\hat{K}_1$ and $\hat{K}_2$ with $\hat{K}_1 < \hat{K}_2$.

a) Either both equilibria are feasible (in the sense that they lie in Region I; no bankruptcy), or both equilibria are infeasible, or only the equilibrium with the larger capital stock, $\hat{K}_2$, is feasible.

b) The equilibrium is either "saddle point" stable in the sense that one eigenvalue is r, the other two eigenvalues are real and symmetric around $r/2$, one being positive and one being negative. Or it is totally unstable: one eigenvalue is r, the other two eigenvalues are either real, positive, and symmetric around $r/2$, or they are conjugate complex with positive real part $r/2$

Proof: Part a) follows from $A(\hat{S}_2) - v\hat{K}_2 < A(\hat{S}_1) - v\hat{K}_1$ where $\hat{S}_i$ is the equilibrium value of the safety expenditures associated with $\hat{K}_i$. Here it is important to remember that S increases with K.

Part b) follows from the above Jacobian matrix (36). Clearly, $\mu_1 = r$ is one of the eigenvalues, and we need only consider the upper left 2×2 submatrix, which we call J_2, and obtain its eigenvalues $\mu_{2,3}$:

$$\mu_{2,3} = \frac{r \pm \sqrt{r^2 - 4\det J_2}}{2} \qquad \text{with} \qquad \det J_2 = -a[r+a] + \frac{R''}{c} + \frac{w}{c\gamma K^2}. \tag{37}$$

If $\det J_2 > 0$, then all eigenvalues $\mu_{2,3}$ are real and positive or they are conjugate complex with positive real part $r/2$. Thus the equilibrium is totally unstable.

If $\det J_2 < 0$, then both eigenvalues $\mu_{2,3}$ are real and μ_2 is positive while μ_3 is negative.

This completes the proof.

4.3.1 Example

Let us now consider again the case where we specify the model functions as in (27) to (29). Then Proposition 3 can be sharpened to:

Proposition 4: Let us consider the case where equation (30) has two roots, and let us denote the corresponding capital stocks by $\hat{K}_1$ and $\hat{K}_2$ with $\hat{K}_1 < \hat{K}_2$.

a) Either both equilibria are feasible (in the sense that they lie in Region I; no bankruptcy), or both equilibria are infeasible, or only the equilibrium with the larger capital stock, $\hat{K}_2$, is feasible.

b) The equilibrium with the larger capital stock is "saddle point" stable in the sense the one eigenvalue is r, the other two eigenvalues are real and symmetric around $r/2$, one being positive and one being negative

c) The equilibrium with the smaller capital stock is totally unstable: one eigenvalue is r, the other two eigenvalues are either real, positive, and symmetric around $r/2$, or the are conjugate complex with positive real part $r/2$

Proof: Part a) follows from Proposition 3a.

In order to prove Part b) , we make use of the functions specified in (27) – (29), and obtain from (37) that

$$\det J_2 = -a[r+a] + \frac{b[b-1]K^{b-2}}{c} + \frac{w}{c\gamma K^2} \tag{38}$$

and therefore the sign of $\det J_2$ equals the sign of

$$G(K) = -c\gamma\alpha[r+a]K^2 + \gamma b[b-1]K^b + w. \tag{39}$$

By comparing (31) and (39), it is straightforward to verify, that functions $F(K)$ and $G(K)$ intersect each other at the value $\tilde{K}$ from (32), and that $F(K) < G(K)$ for $0 < K < \tilde{K}$, while $F(K) > G(K)$ for $K > \tilde{K}$. In the case of two equilibria (see Fig. 7), $G(\tilde{K}) = F(\tilde{K}) < 0$. Furthermore G is easily seen to be decreasing and $G(0) = w > 0$.

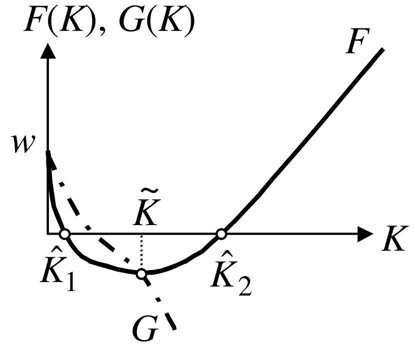


Figure* 10*: The values of G in the two equilibria.

Hence for the equilibrium with the smaller capital stock $\hat{K}_1$, we have $\operatorname{sgn}(\det J_2) = \operatorname{sgn} G(\hat{K}_1) = +1$, while for the equilibrium with the larger capital stock $\hat{K}_2$, we obtain $\operatorname{sgn}(\det J_2) = \operatorname{sgn} G(\hat{K}_2) = -1$. This completes the proof.

4.4 Stability behavior in a numerical example

We have performed numerical calculations in order to illustrate the results obtained so far. For instance, the parameters were chosen as:

$c = 1$, $b = 0.5$, $\alpha = 5$, $\gamma = 0.08$, $r = 0.1$, $a = 0.1$, $w = 0.05$, $\mu = 0.1$, $\beta = 0.5$, $v = 5$.

For these, there are two equilibria in Region I (no bankruptcy):

$\hat{K}_1 = 1.97848$ with eigenvalues 0.1, $0.05 \pm 0.21756i$, and
$\hat{K}_2 = 4.91917$ with eigenvalues 0.1, -0.08995, and 0.18995.

In Figure 11, we have sketched this situation in the (K, I, λ_2)-space. It is seen that the one-dimensional stable manifold of the larger equilibrium $\hat{K}_2$ emanates from the totally unstable smaller equilibrium $\hat{K}_1$ in form of a spiral, i.e. two eigenvalues are conjugate complex there.

In analogy to the case with one state variable (see, e.g. Section 4.5.2 in Feichtinger and Hartl, 1986), it seems reasonable that there is a Skiba-point near $\hat{K}_1$ so that for values of K larger than this threshold, the optimal solution will converge against $\hat{K}_2$ (following the stable manifold). Most likely, for values of K smaller than this threshold, the firm will leave the non-bankruptcy region. A further analysis of the dynamic system in the other two regions is required in order to find out what the solution then looks like. It is expected that the firm will go out of business, i.e. $K \to 0$.

Another possibility would be that there is no equilibrium, in which case the firm will always leave the non-bankruptcy region. A third possibility is the hairline case with only one equilibrium which has, however, no economic meaning.

Figure* 11*: The stability of the two equilibria.

5 Final Remarks

In the objective (3) we have implicitly assumed that the firm has to pay for the final damage by which it goes bankrupt. This could be seen as the objective of the manager of the firm who has to take this final damage into consideration. The reason is that a large final damage is very bad for his reputation.

An alternative formulation would be to consider the shareholders' value, which is almost identical with (3) except for the final damage expression. If the firm goes bankrupt by a large accident, then the shareholders simply refuse to pay for it. The corresponding objective then would be

$$\max_{S,I} \int_0^\infty e^{-rt} \left[R(K) - wS - C(I) - \mu K A(S) \left[1 - H\left(A(S) - vK \right) \right] \right] \left[1 - F \right] dt \, , \tag{40}$$

i.e. the damage A is only paid for when the firm does not go bankrupt by this accident.

Using (5) this can also be written as

$$\max_{S,I} \int_0^\infty e^{-rt} \left[R(K) - wS - C(I) - \mu K A(S) \right] \left[1 - F \right] dt + \int_0^\infty e^{-rt} A(S) \dot{F} \, dt \tag{41}$$

where the last term is a correction factor saying that at the damage A(S) does not count if the firm goes bankrupt.

Clearly some of the calculations in Section 4 will change. For instance, it makes no sense for the firm to spend money on safety, if the firm goes bankrupt anyway after the occurrence of an accident, as it is the case in Region III in figure 3. This implies that S can only be zero if it falls below S_1. See Figures 4 and 5.

On the other hand the optimal long run investment behavior is the same in both models provided that the equilibrium occurs in the non-bankruptcy region. This means that all calculations and results of Section 4.2 carry over to the model with objective (40).

6 References

[1] **Beard,T.R.,** 1990, Bankruptcy and care choice, *Rand Journal of Economics*, **21**, 626-634.

[2] **Brander, J., Spencer, B.,** 1989, Moral hazard and limited liability: Implication for the theory of the firm, *International Economic Review*, **30**, 833-849.

[3] **Feichtinger, G., Hartl, R.F.,** 1986, *Optimale Kontrolle ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, de Gruyter, Berlin.

[4] **Van Hilten, O., Kort, P.M., van Loon, P.J.J.M.,** 1993, *Dynamic Policies of the Firm*, Springer, Berlin.

[5] **Veld, Klaas van 't**: The judgment proof opportunity, Working Paper No. 823, Dept. of Agricultural and Resource Economics, University of California at Berkeley, 1997.